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Noetherian approximation of algebraic spaces and stacks

David Rydh [1]

KTH Royal Institute of Technology, Department of Mathematics, SE-100 44 Stockholm, Sweden

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ABSTRACT

We show that every scheme (resp. algebraic space, resp. algebraic stack) that is quasi-compact with quasi-finite diagonal can be approximated by a noetherian scheme (resp. algebraic space, resp. stack). More generally, we show that any stack which is étale-locally a global quotient stack can be approximated. Examples of applications are generalizations of Chevalley's, Serre's and Zariski's theorems and Chow's lemma to the non-noetherian setting. We also show that every quasi-compact algebraic stack with quasi-finite diagonal has a finite generically flat cover by a scheme.

Contents

E-mail address: dary@math.kth.se.

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Introduction

Let A be a commutative ring and let M be an A-module. Then A is the direct limit of its subrings that are finitely generated as $\mathbb{Z}$-algebras and M is the direct limit of its finitely generated A-submodules. Thus, any affine scheme X is an inverse limit of affine schemes of finite type over $\operatorname{Spec}\mathbb{Z}$ and every quasi-coherent sheaf on X is a direct limit of quasi-coherent sheaves of finite type.

The purpose of this article is to give similar approximation results for schemes, algebraic spaces and stacks, generalizing earlier results for schemes by R.W. Thomason and T. Trobaugh [28, App. C]. We show, for example, that every quasi-compact and quasi-separated Deligne–Mumford stack X can be written as an inverse limit of Deligne–Mumford stacks X_λ of finite type over $\operatorname{Spec}\mathbb{Z}$. Such results are sometimes known as "absolute approximation" [28, App. C] in contrast with the "standard limit results" [9, §8] which are relative: given an inverse limit $X = \varprojlim_\lambda X_\lambda$, describe *finitely presented* objects over X in terms of finitely presented objects over X_λ for sufficiently large λ.

We say that an algebraic stack X is of *global type* if étale-locally X is a global quotient stack, cf. Section 2 for a precise definition. Examples of stacks of global type are quasi-compact and quasi-separated schemes, algebraic spaces, Deligne–Mumford stacks and algebraic stacks with quasi-finite (and locally separated) diagonals. For convenience, we also introduce the notion of approximation type. Every stack of global type is of approximation type (Proposition 2.10).

The main result of this paper, Theorem D, is that any stack of approximation type can be approximated by a noetherian stack. More generally, if $X \to S$ is a morphism between stacks of approximation type, then X is an inverse limit of finitely presented stacks over S.

The primary application of the approximation theorem is the elimination of noetherian, excellency and finiteness hypotheses. When eliminating noetherian hypotheses in statements about finitely presented morphisms $X \to Y$ which are *local* on Y, the basic affine approximation result referred to in the beginning is sufficient, cf. the standard limit results in [9, §8] and Appendix B. For *global* problems, it is crucial to have Theorem D. Examples of such applications, including generalizations of Chevalley's, Serre's and Zariski's theorems and Chow's lemma, are given in Section 8. Although this paper is written with stacks in mind, most of the applications in Section 8 are