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Algebraic study on Cameron–Walker graphs

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ABSTRACT

Let G be a finite simple graph on $[n]$ and $I(G) \subset S$ the edge ideal of G, where $S = K[x_1, \dots, x_n]$ is the polynomial ring over a field K. Let $m(G)$ denote the maximum size of matchings of G and $im(G)$ that of induced matchings of G. It is known that $im(G) \leq \mathrm{reg}(S/I(G)) \leq m(G)$, where $\mathrm{reg}(S/I(G))$ is the Castelnuovo–Mumford regularity of $S/I(G)$. Cameron and Walker succeeded in classifying the finite connected simple graphs G with $im(G) = m(G)$. We say that a finite connected simple graph G is a Cameron–Walker graph if $im(G) = m(G)$ and if G is neither a star nor a star triangle. In the present paper, we study Cameron–Walker graphs from a viewpoint of commutative algebra. First, we prove that a Cameron–Walker graph G is unmixed if and only if G is Cohen–Macaulay and classify all Cohen–Macaulay Cameron–Walker graphs. Second, we prove that there is no Gorenstein Cameron–Walker graph. Finally, we prove that every Cameron–Walker graph is sequentially Cohen–Macaulay.

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Introduction

Recently, edge ideals of finite simple graphs have been studied by many authors from viewpoints of computational commutative algebra and combinatorics; see [8,14,17], and their references.

Let $[n] = \{1, \ldots, n\}$ be a vertex set and G a finite simple graph on $[n]$ with $E(G)$ its edge set. (A simple graph is a graph with no loop and no multiple edge.) Let $S = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over a field K. The *edge ideal* [10, p. 156] of G is the monomial ideal $I(G)$ of S generated by those monomials $x_i x_j$ with $\{i, j\} \in E(G)$, viz.,

$$I(G) = \big(x_i x_j : \{i, j\} \in E(G)\big) \subset S.$$

One of the research topics on $I(G)$ is the computation of the Castelnuovo–Mumford regularity $\mathrm{reg}(S/I(G))$ [10, p. 48] of $S/I(G)$ in terms of the invariants of G.

Recall that a subset M of $E(G)$ is a *matching* of G if, for e and e' belonging to M with $e \neq e'$, one has $e \cap e' = \emptyset$. The *matching number* $m(G)$ of G is the maximum size of matchings of G. A matching M of G is called an *induced matching* of G if, for e and e' belonging to M with $e \neq e'$, there is no $f \in E(G)$ with $f \cap e \neq \emptyset$ and $e' \cap f \neq \emptyset$. Let $im(G)$ denote the maximum size of induced matchings of G.

For example, if $G = K_n$, a complete graph on $[n]$, then $im(G) = 1$ and $m(G) = \lfloor n/2 \rfloor$. If $G = K_{m,n}$ $(m \leq n)$, a complete bipartite graph with vertex partition $[n] \sqcup [m]$, then $im(G) = 1$ and $m(G) = m$. If G is the Petersen graph, then $im(G) = 3$ and $m(G) = 5$.

It is known ([11, Lemma 2.2] and [9, Theorem 6.7]) that

$$im(G) \leq \mathrm{reg}\big(S/I(G)\big) \leq m(G).$$

One has $\mathrm{reg}(S/I(G)) = im(G)$ for, e.g., chordal graphs, unmixed bipartite graphs and sequentially Cohen–Macaulay bipartite graphs ([9,12,15]; see also [7,13,19,20]).

Cameron and Walker [3, Theorem 1] gave a classification of the finite connected simple graphs G with $im(G) = m(G)$, although there is a mistake; see Remark 0.1 below. By modifying their result slightly, we see that a finite connected simple graph G satisfies $im(G) = m(G)$ if and only if G is one of the following graphs:

- a star;
- a star triangle;
- a finite graph consisting of a connected bipartite graph with vertex partition $[n] \sqcup [m]$ such that there is at least one leaf edge attached to each vertex $i \in [n]$ and that there may be possibly some pendant triangles attached to each vertex $j \in [m]$.

Here a star triangle is a graph joining some triangles at one common vertex, e.g., the graph on $\{1, \ldots, 7\}$ with the edges $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 4\}$, $\{1, 5\}$, $\{4, 5\}$, $\{1, 6\}$,