Journal of Algebra 422 (2015) 318–333

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Finite soluble groups with metabelian centralizers

Carlo Casolo [a], Enrico Jabara [b,*]

[a] *Dipartimento di Matematica "U. Dini", Università di Firenze, Viale Morgagni, 67/A, 50134 Firenze, Italy*

[b] *Dipartimento di Filosofia e Beni Culturali, Università di Ca' Foscari – Venezia, Dorsoduro, 3484/D, 30123 Venezia, Italy*

A R T I C L E   I N F O

Article history:
Received 11 June 2014
Available online 16 October 2014
Communicated by Gernot Stroth

MSC:
20D10
20F14

Keywords:
Group theory
Finite soluble groups
Centralizers

A B S T R A C T

If G is a finite soluble group in which the centralizer of every non-trivial element is metabelian (or nilpotent-by-abelian), then G has derived length at most 4 (respectively, the third term of the derived series is nilpotent).

1. Introduction

This paper is part of a program whose general aim is to describe, in a quantitative way when possible, the implications that, in finite soluble groups, structural conditions on the centralizers of non-trivial elements have on the structure of the whole group. That, in principle, these implications should exist and be rather decisive is suggested by many known results: we mention, for instance, a theorem of Isaacs [2] which says that in a finite soluble group there always exists a non-trivial element the order of whose

* Corresponding author.
E-mail addresses: casolo@math.unifi.it (C. Casolo), jabara@unive.it (E. Jabara).

http://dx.doi.org/10.1016/j.jalgebra.2014.09.032

centralizer exceeds the square root of the order of the group. However, the classical example which is the starting motivation for the present article is that of CA-groups: that is groups in which the centralizers of non-trivial elements are abelian. While the great importance of this class lies in the stimulus its study gave in developing techniques for simple groups, it is also interesting, although an easy exercise, to recall the fact that a soluble CA-group is either abelian or a metabelian Frobenius group. In particular, a soluble CA-group has derived length at most 2. Thus, we ask whether bounds exist (and can be determined with a certain precision) for the derived length of soluble groups in which a bound on the derived length of the centralizers is imposed.

Before proceeding further, let us agree that all groups considered in this paper are finite. If G is a group, we denote by $F(G)$ the Fitting radical of G, and for $n \geq 1$, by $F_n(G)$ the n-th term of the Fitting series of G (i.e. $F_1(G) = F(G)$ and $F_{n+1}(G)/F_n(G) = F(G/F_n(G))$ for $n \geq 1$). Also, we denote by $d(G)$ and $h(G)$, respectively, the derived length and the Fitting length of G.

For a positive integer d, we denote by CA_d the class of all groups in which the centralizer of any non-trivial element is soluble of derived length at most d. Of course, conditions on the centralizers in group actions already inspired a quite well developed and important topic in literature, specially when imposed to the centralizers of non-trivial elements of the acting group in coprime action, and it is predictable that some of the results established in the coprime action case may play a key role in our more specific perspective. In fact, as a rather immediate application of one of these (due to A. Turull), we will first provide a general bound for the derived lengths.

Theorem 1. *Let $d \geq 1$ and G be a soluble group in CA_d. Then $d(G) \leq d(3d + 2)$ and $h(G) \leq 3d + 2$.*

Even though the quadratic bound in Theorem 1 might appear similar to the quadratic bound in Isaacs's theorem mentioned at the beginning, we believe that this is definitely too large, and ask whether a linear bound for $d(G)$ exists. However, we do not address here this general question any further: the main purpose of this paper is, indeed, to determine the exact bounds both for $h(G)$ and $d(G)$ for soluble groups in CA_2. The main result is then the following

Theorem 2. *Let G be a soluble CA_2-group; then $d(G) \leq 4$.*

This bound is best possible, in the sense that there exist soluble CA_2-groups whose Fitting length is 4 (see Example B at the end of this section). In fact, with a little more effort, we may prove similar bounds allowing the centralizer to be somehow larger. More precisely, let CN' denote the class of all groups G in which, for every $1 \neq x \in G$, the derived subgroup of the centralizer $C_G(x)$ is nilpotent. Then

Theorem 3. *Let G be a soluble CN'-group; then $G^{(3)}$ is nilpotent.*