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# The Witt ring of a curve with good reduction over a non-dyadic local field

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ABSTRACT

In this work, we present a generalization to varieties and sheaves of the fundamental ideal of the Witt ring of a field by defining a sheaf of fundamental ideals $\tilde{I}$ and a sheaf of Witt rings $\tilde{W}$ in the obvious way. The Milnor conjecture then relates the associated graded of $\tilde{W}$ to Milnor K-theory and so allows the classical invariants of a bilinear space over a field to be extended to our setting using étale cohomology. As an application of these results, we calculate the Witt ring of a smooth curve with good reduction over a non-dyadic local field.



## 1. Introduction

The goals of this work are twofold. First, we present a generalization to the category of varieties and sheaves of the fundamental ideal of the Witt ring and of some techniques associated with this ideal which have proven useful in describing the Witt ring of a field.

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Second, we calculate the Witt ring of a smooth projective curve with good reduction over a non-dyadic local field as an application of these techniques.

Much work has been done toward generalizing the Witt ring, its fundamental ideal, and the connections between the latter, K-theory, and cohomology. Calculations of Witt groups and rings using such connections are seen, for example, in Carmena's work on complex surfaces [3], Sujatha's on real projective surfaces [12], and Parimala's on affine three folds [11]. [1] is a good recent literature review on generalization of the Milnor conjecture to the categories of rings and schemes. Unless otherwise specified, we will use definitions presented by Knebusch [9], whose work forms much of the basis for study of Witt rings of varieties and schemes.

Our approach is to define a sheaf of fundamental ideals $\tilde{I}$ and a sheaf of Witt rings $\tilde{W}$. For low dimensional cases, the global sections $\mathbb{I}^n := \Gamma(X, \tilde{I}^n)$ form a filtration of the Witt ring $W(X)$ and we can study the Witt ring via the quotients $\mathbb{I}^n/\mathbb{I}^{n+1}$. We will show that $\tilde{I}^n/\tilde{I}^{n+1}$ is isomorphic to the Zariski sheafification of étale cohomology with $\mu_2$ coefficients (this follows from the Milnor conjecture and most of the work has already been done by Kerz, Milnor, Orlov, Vishik, and Voevodsky, among others). We then show that $\tilde{W}/\tilde{I}$, $\tilde{I}/\tilde{I}^2$, and $\tilde{I}^2/\tilde{I}^3$ are isomorphic to the Zariski sheaves $\mathbb{Z}/2\mathbb{Z}$, $\mathcal{O}^\times/\mathcal{O}^{\times 2}$, and ${}_2Br$ respectively and that the classes of elements in $W(X)/\mathbb{I}$, $\mathbb{I}/\mathbb{I}^2$, and $\mathbb{I}^2/\mathbb{I}^3$ are determined by rank, signed discriminant, and Witt invariant, respectively.

For a smooth geometrically connected projective curve $C$ with good reduction $C_k$ over a non-dyadic local field with uniformizing parameter $\pi$ we can show that $\mathbb{I}^n/\mathbb{I}^{n+1}$ is trivial when $n > 2$. Thus, using the generalizations outlined above, we can describe each element of the Witt ring in terms of its rank, signed discriminant, and Witt invariant. We may equate each space of odd rank to $\langle -1 \rangle$ in $W(X)/\mathbb{I}$ and each even rank space to $\langle 1, \delta \rangle$ in $\mathbb{I}/\mathbb{I}^2$ where $\delta$ is the signed discriminant. It remains to describe the contribution from $\mathbb{I}^2/\mathbb{I}^3$, which is determined by the class of the Clifford algebra in the order 2 part of the Brauer group. We show that, over the function field of $C$, ${}_2Br(C)$ consists of distinct "quaternions" $(\frac{\xi, \pi}{k(C)})$, allowing us to equate each member of $\mathbb{I}^2$ to the four-dimensional form $\langle 1, -\xi, -\pi, \pi\xi \rangle$ in $\mathbb{I}^2/\mathbb{I}^3$. In this way, we can explicitly represent each element of the Witt ring of a sum of line bundles and prove that $W(C) \simeq W(C_k)[\mu_2]$, which nicely generalizes the classical result for the Witt ring of a local field [10].

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## 2. The fundamental ideal

Define $\tilde{W}$ and $\tilde{I}$ to be the sheaves associated to the presheaves $W(-) : U \mapsto W(\mathcal{O}_X(U))$ and $I(-) : U \mapsto I(W(U))$ respectively. The sheaves $\tilde{I}^n$ form a filtration of $\tilde{W}$ and the global sections $\mathbb{I}^n := \Gamma(X, \tilde{I}^n)$ form a filtration of $\Gamma(X, \tilde{W})$.

Assuming that $X$ is proper, smooth, and geometrically connected and that $dim(X) \leq 3$, we first recall a result [2] that establishes $W(X) = \Gamma(X, \tilde{W})$ and so the $\mathbb{I}^n$ form a filtration of $W(X)$.