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The representation theory of the exceptional Lie superalgebras $F(4)$ and $G(3)$

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ABSTRACT

This paper is a resolution of three related problems proposed by Yu.I. Manin and V. Kac for the so-called *exceptional* Lie superalgebras $F(4)$ and $G(3)$. The first problem posed by Kac (1978) is the problem of finding character and superdimension formulae for the simple modules. The second problem posed by Kac (1978) is the problem of classifying all indecomposable representations. The third problem posed by Manin (1981) is the problem of constructing the *superanalogue* of Borel–Weil–Bott theorem.

1. Introduction

After classifying all finite-dimensional simple Lie superalgebras over $\mathbb{C}$ in 1977, V. Kac proposed the problem of finding character and superdimension formulae for the simple modules (see [8]). The first main result in this paper is solving this problem in full for the so-called *exceptional* Lie superalgebras $F(4)$ and $G(3)$.

The next problem, also posed by V. Kac in 1977, is the problem of classifying all indecomposable representations of classical Lie superalgebras (see [8]). Here we settle it as follows. For the exceptional Lie superalgebras $F(4)$ and $G(3)$, we describe the

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blocks up to equivalence and find the corresponding *quivers*, which gives a full solution of this problem. We show that the blocks of atypicality 1 are *tame*, which together with Serganova's results for other Lie superalgebras proves a conjecture by J. Germoni.

In the geometric representation theory of Lie algebras, the Borel–Weil–Bott (BWB) theorem (see Theorem 2.7) plays a crucial role. This theorem describes how to construct families of representations from sheaf cohomology groups associated to certain vector bundles. It was shown by I. Penkov, that this theorem is not true for Lie superalgebras. In 1981, Yu.I. Manin proposed the problem of constructing a *superanalogue* of BWB theorem. The first steps towards the development of this theory were carried out by I. Penkov in [15]. One of my results (see Theorem 2.8 and Theorem 2.9) is an analogue of BWB theorem for the exceptional Lie superalgebras $F(4)$ and $G(3)$ for dominant weights.

The *basic classical* Lie superalgebras that are not Lie algebras are:

(i) the series $\mathfrak{sl}(m|n)$ and $\mathfrak{osp}(m|n)$;

(ii) the exceptional Lie superalgebras $F(4)$ and $G(3)$; and

(iii) the family of exceptional Lie superalgebras $D(2,1;\alpha)$.

In [8], Kac introduced the notions of *typical* and *atypical* irreducible representations. He classified the finite-dimensional irreducible representations for basic classical Lie superalgebras using highest weights and induced module constructions similar to Verma module constructions for simple Lie algebras. In [10], he found character formulae similar to the Weyl character formula for typical irreducible representations.

The study of atypical representations has been difficult and has been studied intensively over the past 40 years. Unlike the typical modules, atypical modules are not uniquely described by their central character. All simple modules with given central character form a *block* in the category of finite-dimensional representations.

The problem of finding characters for simple finite-dimensional $\mathfrak{gl}(m|n)$-modules has been solved using a geometric approach by V. Serganova in [19] and [18], and later J. Brundan in [1] found characters using algebraic methods and computed extensions between simple modules.

More recently, this problem has been solved for all infinite series of basic classical Lie superalgebras in [7] by C. Gruson and V. Serganova. They compute the characters of simple modules using Borel–Weil–Bott theory and generalizing a combinatorial method of weight and cap diagrams developed first by Brundan and Stroppel for $\mathfrak{gl}(m|n)$ case.

In [4], J. Germoni solves Kac's problems for $D(2,1;\alpha)$. He also studies the blocks for $G(3)$ using different methods, but the problems by Kac and Manin remained open for $G(3)$ and $F(4)$. I solve the above problems for $G(3)$ and $F(4)$, generalizing the methods of [7].

Strategies used for the exceptional Lie superalgebras $F(4)$ and $G(3)$: To solve the aforementioned problems of V. Kac and Yu.I. Manin for the superalgebras $F(4)$ and $G(3)$,