Journal of Algebra 419 (2014) 318–331

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Distractions of Shakin rings

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A R T I C L E   I N F O

Article history:
Received 21 August 2013
Communicated by Steven Dale Cutkosky

MSC:
primary 13A02, 13D02
secondary 13P10, 13P20

Keywords:
Eisenbud–Green–Harris Conjecture
Embeddings of Hilbert functions
Distractions
Piecewise lex-segment ideals

A B S T R A C T

We study, by means of embeddings of Hilbert functions, a class of rings which we call Shakin rings, i.e. quotients $K[X_1, \dots, X_n]/\mathfrak{a}$ of a polynomial ring over a field K by ideals $\mathfrak{a} = L + P$ which are the sum of a piecewise lex-segment ideal L, as defined by Shakin, and a pure powers ideal P. Our main results extend Abedelfatah's recent work on the Eisenbud–Green–Harris Conjecture, Shakin's generalization of Macaulay and Bigatti–Hulett–Pardue Theorems on Betti numbers and, when $\mathrm{char}(K) = 0$, Mermin–Murai Theorem on the Lex-Plus-Power inequality, from monomial regular sequences to a larger class of ideals. We also prove an extremality property of embeddings induced by distractions in terms of Hilbert functions of local cohomology modules.

Introduction

Hilbert functions are an important object of study in Commutative Algebra and Algebraic Geometry, since they encode several fundamental invariants of varieties and their coordinate rings such as dimension and multiplicity. A notable result is due to

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[1] The work of the first author was supported by a grant from the Simons Foundation (209661 to G.C.).

http://dx.doi.org/10.1016/j.jalgebra.2014.05.008
0021-8693/

Macaulay [19], who provided a characterization of the numerical functions which are Hilbert functions of standard graded algebras by means of lexicographic (or lex-segment) ideals. Later, Kruskal and Katona [18,17] completely characterized the numerical sequences which are f-vectors of abstract simplicial complexes, thus establishing a remarkable analogue of the Macaulay Theorem in Algebraic and Extremal Combinatorics which can be rephrased in terms of Hilbert functions of graded quotients of algebras defined by monomial regular sequences of pure quadrics.

One of the most relevant open problems in the study of Hilbert functions is a conjecture, due to Eisenbud, Green and Harris [11,12], which aims at extending Kruskal–Katona Theorem (and the subsequent generalization of Clements and Lindström [10]) to a larger class of objects, namely coordinate rings of complete intersections, and obtaining in this way a strong generalization of the Cayley–Bacharach Theorem for projective plane cubic curves. The Eisenbud–Green–Harris Conjecture predicts that all Hilbert functions of homogeneous ideals of $R = A/\mathfrak{a}$, where A is a polynomial ring over a field K and $\mathfrak{a}$ is an ideal of A generated by a homogeneous regular sequence, are equal to those of the images of some lex-segment ideals of A in the quotient ring A/P, where P is generated by a certain regular sequence of pure powers of variables.

This conjecture, which has been solved in some cases [1,6,8–10,13], renewed a great deal of interest in understanding and eventually classifying Hilbert functions of quotients of standard graded algebras $R = A/\mathfrak{a}$, where A is a polynomial ring over a field K and $\mathfrak{a}$ is a fixed homogeneous ideal of A, in terms of specific properties of $\mathfrak{a}$.

In recent years, Mermin, Peeva and their collaborators started a systematic investigation of rings $R = A/\mathfrak{a}$ for which all the Hilbert functions of homogeneous ideals are obtained by Hilbert functions of images in R of lex-segment ideals. They called these rings Macaulay-lex [14,20–25]. Two typical examples of such rings are the polynomial ring A and the so-called Clements–Lindström rings, i.e. rings of the type $R = A/P$ where $P = (X_1^{d_1}, \ldots, X_r^{d_r})$ and $d_1 \leq \cdots \leq d_r$.

In a polynomial ring A, among all the graded ideals with a fixed Hilbert function, the lex-segment ideal enjoys several extremal properties. We summarize some of them in three categories.

1. Lex-segment ideals are the ones with the largest number of minimal generators. Precisely, for a fixed Hilbert function and for every j, the lex-segment ideal maximizes the values of $\beta_{0j}^A(-)$, and hence the value of $\beta_0^A(-)$. This fact is a consequence of Macaulay Theorem.
2. More generally, by theorems of Bigatti, Hulett [2,16] (when $\mathrm{char}(K) = 0$) and [26], for every i, j the lex-segment ideal also maximizes the graded Betti numbers $\beta_{ij}^A(-)$.
3. Finally, by [27], the lex-segment ideal maximizes the Hilbert functions of the local cohomology modules of $A/(-)$; precisely, for every i and j it maximizes $\dim_K H_{\mathfrak{m}}^i(A/-)_j$ where $\mathfrak{m}$ is the homogeneous maximal ideal of A.