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Periodicity of Betti numbers of monomial curves

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ABSTRACT

Let K be an arbitrary field. Let $\mathbf{a} = (a_1 < \cdots < a_n)$ be a sequence of positive integers. Let $C(\mathbf{a})$ be the affine monomial curve in $\mathbb{A}^n$ parametrized by $t \to (t^{a_1}, ..., t^{a_n})$. Let $I(\mathbf{a})$ be the defining ideal of $C(\mathbf{a})$ in $K[x_1, ..., x_n]$. For each positive integer j, let $\mathbf{a} + j$ be the sequence $(a_1 + j, ..., a_n + j)$. In this paper, we prove the conjecture of Herzog and Srinivasan saying that the Betti numbers of $I(\mathbf{a}+j)$ are eventually periodic in j with period $a_n - a_1$. When j is large enough, we describe the Betti table for the closure of $C(\mathbf{a} + j)$ in $\mathbb{P}^n$.

1. Introduction

Let K denote an arbitrary field. Let R be the polynomial ring $K[x_1, ..., x_n]$. Let $\mathbf{a} = (a_1 < \cdots < a_n)$ be a sequence of positive integers. The sequence $\mathbf{a}$ gives rise to a monomial curve $C(\mathbf{a})$ whose parametrization is given by $x_1 = t^{a_1}, ..., x_n = t^{a_n}$. Let $I(\mathbf{a})$ be the defining ideal of $C(\mathbf{a})$. For each positive integer j, let $\mathbf{a} + j$ be the sequence $(a_1 + j, ..., a_n + j)$. In this paper, we consider the behavior of the Betti numbers of the defining ideals $I(\mathbf{a} + j)$ and their homogenizations $\bar{I}(\mathbf{a} + j)$ for positive integers j.

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For each finitely generated R-module M and each integer i, let

$$\beta_i(M) = \dim_K \operatorname{Tor}_i^R(M, K)$$

be the i-th total Betti number of M. The following conjecture was communicated to us by Herzog and Srinivasan.

Conjecture 1 *(Herzog–Srinivasan). The Betti numbers of $I(\mathbf{a}+j)$ are eventually periodic in j with period $a_n - a_1$.*

In general, the problem of finding defining equations of monomial curves is difficult. For example, in [1], Bresinsky gave an example of a family of monomial curves in $\mathbb{A}^4$ whose numbers of minimal generators of the defining ideals are unbounded. Recently, in the case $n \leq 3$, Conjecture 1 was proven by Jayanthan and Srinivasan in [7]. In the case when $\mathbf{a}$ is an arithmetic sequence, Conjecture 1 was proven by Gimenez, Sengupta and Srinivasan in [6]. In this paper, we prove the conjecture in full generality:

Theorem 1.1. *The Betti numbers of $I(\mathbf{a} + j)$ are eventually periodic in j with period $a_n - a_1$.*

To prove Theorem 1.1 we first prove the eventual periodicity in j for total Betti numbers of the homogenization $\bar{I}(\mathbf{a} + j)$, and then prove the equalities for total Betti numbers of $I(\mathbf{a} + j)$ and $\bar{I}(\mathbf{a} + j)$ when $j \gg 0$.

To simplify notation, for each i, $1 \leq i \leq n$, let $b_i = a_n - a_i$. Note that if f is homogeneous then $f \in I(\mathbf{a})$ if and only if $f \in I(\mathbf{a} + j)$ for all j. Denote by $J(\mathbf{a})$ the ideal generated by homogeneous elements of $I(\mathbf{a})$. In general, for each finitely generated graded R-module M, $\operatorname{Tor}_i^R(M, K)$ is a finitely generated graded module for each i. Let

$$\beta_{ij}(M) = \dim_K \operatorname{Tor}_i^R(M, K)_j$$

be the i-th graded Betti number of M in degree j. Moreover, let

$$\operatorname{reg} M = \sup_{i,j}\{j - i : \beta_{ij} \neq 0\}$$

be the Castelnuovo–Mumford regularity of M.

Let x_0 be a homogenizing variable. In Proposition 3.3, we prove that when $j > b_1(n + \operatorname{reg} J(\mathbf{a}))$, each binomial in $\bar{I}(\mathbf{a} + j)$ involving x_0 has degree greater than $n + \operatorname{reg} J(\mathbf{a})$. Thus the Betti table of $\bar{I}(\mathbf{a} + j)$ separates into two parts. One part is the Betti table of $J(\mathbf{a})$ which lies in degree at most $n + \operatorname{reg} J(\mathbf{a})$. The other part lies in degree larger than $n + \operatorname{reg} J(\mathbf{a})$. We call the part of Betti table of $\bar{I}(\mathbf{a} + j)$ lying in degree larger than $n + \operatorname{reg} J(\mathbf{a})$ the high degree part. We will prove that when $j \gg 0$, the Betti table of $\bar{I}(\mathbf{a} + j + b_1)$ is obtained from the Betti table of $\bar{I}(\mathbf{a} + j)$ by shifting the high degree part of $\bar{I}(\mathbf{a} + j)$ (see Theorem 4.6).