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On shift dynamics for cyclically presented groups

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ABSTRACT

A group defined by a finite presentation with cyclic symmetry admits a shift automorphism that is periodic and word-length preserving. It is shown that if the presentation is combinatorially aspherical and orientable, in the sense that no relator is a cyclic permutation of the inverse of any of its shifts, then the shift acts freely on the non-identity elements of the group presented. For cyclic presentations defined by positive words of length at most three, the shift defines a free action if and only if the presentation is combinatorially aspherical and the shift itself is fixed point free if and only if the group presented is infinite.

1. Cyclically presented groups

A celebrated theorem of J. Thompson [28] on Frobenius kernels states that any finite group that possesses a fixed point free automorphism of prime order must be nilpotent. A combinatorial template for a group with a periodic automorphism is one that admits a **cyclic presentation** of the form

$$\mathcal{P}_n(w) = \bigl(x_0, \ldots, x_{n-1} : w, \theta(w), \ldots, \theta^{n-1}(w)\bigr)$$

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where n is a positive integer, $w = w(x_0, \ldots, x_{n-1})$ is an arbitrary word in the generators $x_0, \ldots, x_{n-1}$, and θ is the **shift** automorphism given by $\theta(x_i) = x_{i+1}$ with subscripts modulo n. The shift θ defines an automorphism of exponent n for the **cyclically presented group** $G_n(w)$ defined by $\mathcal{P}_n(w)$. A fundamental problem in the study of cyclic presentations is to determine whether $G_n(w)$ is finite, and if so, to determine its structure. Thompson's theorem has the following consequence related to the dynamics of the shift action by the cyclic group C_n of order n: *If $G_n(w)$ is finite and C_n acts freely via the shift, then $G_n(w)$ is nilpotent.* This is because freeness of the action is enough to guarantee that $G_n(w)$ has a fixed point free automorphism of prime order, so Thompson's theorem applies. For example, the group $G_5(x_0x_2x_1^{-1})$ is the binary icosahedral group [25, Section 5], which is perfect, and so its shift must have a nonidentity fixed point. Further examples are given in [12, Theorem B], which provides an infinite family of finite non-nilpotent cyclically presented groups. Section 5 below shows how to identify explicit fixed points for the shift on these groups.

The main general result of this paper provides sufficient conditions under which the shift determines a free action on the nonidentity elements of a cyclically presented group. The principal condition is **combinatorial asphericity**; a working definition from [5] is presented in Proposition 3.1 below. A cyclic presentation $\mathcal{P}_n(w)$ is **orientable** if w is not a cyclic permutation of the inverse of any of its shifts. It turns out that $\mathcal{P}_n(w)$ is not orientable if and only if $n = 2m$ is even and there is a word $u = u(x_0, \ldots, x_{n-1})$ such that $w = u\theta^m(u)^{-1}$ (Lemma 3.6). Note that u is fixed by $\theta^m \in \mathrm{Aut}(G_{2m}(u\theta^m(u)^{-1}))$.

Theorem A. *If $\mathcal{P}_n(w)$ is orientable and combinatorially aspherical, then C_n acts freely via the shift on the nonidentity elements of $G_n(w)$.*

Remark. It can happen that $G_n(w)$ is trivial, in which case $\mathcal{P}_n(w)$ is combinatorially aspherical and orientable. See [10,11] for the current state of this problem.

Theorem A says that when the cyclic presentation $\mathcal{P}_n(w)$ is orientable and combinatorially aspherical, the orbits of the shift partition the nonidentity elements of $G_n(w)$ into pairwise disjoint n-cycles. In particular, θ and all of its powers are fixed point free and θ has order n in the automorphism group of $G_n(w)$. (Compare e.g. [4, Corollary 3.2], [19, Theorem 2].) Another way to view this is that if some power of the shift has a nonidentity fixed point, then the presentation $\mathcal{P}_n(w)$ must support an interesting spherical diagram [5]. Perhaps one could prove Theorem A in this way, but that is not the path followed here. Instead, the proof of Theorem A employs the cohomology theory of aspherical relative presentations [3] applied to the split extension $E_n(w) = G_n(w) \rtimes_\theta C_n$ (Theorem 3.4).

Now $E_n(w)$ admits a presentation of the form $(a, x : a^n, W)$ and $G_n(w)$ is the kernel of a retraction of $E_n(w)$ onto C_n. Depending on n and w, there can be several different retractions $\nu^f : E_n(w) \to C_n$, each determined by the image $\nu^f(x) = a^f \in C_n$. Section 2 outlines a Reidemeister–Schreier rewriting process $\rho^f(W)$ to show that the kernel of such