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Direct products and elementary equivalence of polycyclic-by-finite groups

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ABSTRACT

Generalizing previous results, we give an algebraic characterization of elementary equivalence for polycyclic-by-finite groups. We use this characterization to investigate the relations between their elementary equivalence and the elementary equivalence of the factors in their decompositions in direct products of indecomposable groups. In particular, we prove that the elementary equivalence $G \equiv H$ of two such groups G, H is equivalent to each of the following properties: (1) $G \times \cdots \times G$ (k times G) $\equiv H \times \cdots \times H$ (k times H) for an integer $k \geq 1$; (2) $A \times G \equiv B \times H$ for two polycyclic-by-finite groups A, B such that $A \equiv B$. It is not presently known if (1) implies $G \equiv H$ for any groups G, H.

1. Characterization of elementary equivalence

In the present paper, we investigate the elementary equivalence between finitely generated groups and the relations between direct products and elementary equivalence for groups. First we give the relevant definitions.

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We say that two groups M, N are *elementarily equivalent* and we write $M \equiv N$ if they satisfy the same first-order sentences in the language which consists of one binary functional symbol.

We say that a group M is *polycyclic* if there exist some subgroups $\{1\} = M_0 \subset \cdots \subset M_n = M$ with M_{i-1} normal in M_i and M_i/M_{i-1} cyclic for $1 \leq i \leq n$. For any properties P, Q, we say that M is *P-by-Q* if there exists a normal subgroup N which satisfies P and such that M/N satisfies Q.

For each group M, we denote by $Z(M)$ the center of M. For each $A \subset M$, we denote by $\langle A\rangle$ the subgroup of M generated by A. We write

$$\begin{aligned}
M' &= \bigl\langle \bigl\{[x,y] \bigm| x,y \in M\bigr\}\bigr\rangle \quad \text{and,} \quad \text{for any } h,k \in \mathbb{N}^*,\\
M'(h) &= \bigl\{[x_1,y_1]\cdots[x_h,y_h] \bigm| x_1,y_1,\ldots,x_h,y_h \in M\bigr\},\\
\bigtimes^k M &= M \times \cdots \times M \quad (k \text{ times } M),\\
M^k &= \bigl\langle \bigl\{x^k \bigm| x \in M\bigr\}\bigr\rangle \quad \text{and}\\
M^k(h) &= \bigl\{x_1^k \cdots x_h^k \bigm| x_1,\ldots,x_h \in M\bigr\}.
\end{aligned}$$

Each of the properties $M' = M'(h)$ and $M^k = M^k(h)$ can be expressed by a first-order sentence. As polycyclic-by-finite groups are noetherian, it follows from [16, Corollary 2.6.2] that, for each polycyclic-by-finite group M and each $k \in \mathbb{N}^*$, there exists $h \in \mathbb{N}^*$ such that $M' = M'(h)$ and $M^k = M^k(h)$. Actually, the result concerning the property $M' = M'(h)$ was first proved in [15].

For each group M and any subsets A, B, we write $AB = \{xy \mid x \in A \text{ and } y \in B\}$. If A, B are subgroups of M and if A or B is normal, then AB is also a subgroup.

For each group M and each subgroup S of M such that $M' \subset S$, we consider the *isolator* $I_M(S) = \bigcup_{k\in\mathbb{N}^*}\{x \in M \mid x^k \in S\}$, which is a subgroup of M. We write $\Gamma(M) = I_M(Z(M)M')$ and $\Delta(M) = I_M(M')$. We also have $\Gamma(M) = I_M(Z(M)\Delta(M))$. If M is abelian, then $\Delta(M)$ is the *torsion subgroup* $\tau(M)$.

For any $h,k \in \mathbb{N}^*$, there exists a set of first-order sentences which, in each group M with $M' = M'(h)$, expresses that $\Gamma(M)^k \subset Z(M)M'$ (resp. $\Delta(M)^k \subset M'$).

During the 2000s, there was a lot of progress in the study of elementary equivalence between finitely generated groups, with the proof of Tarski's conjecture stating that all free groups with at least two generators are elementarily equivalent (see [5] and [17]).

Anyway, examples of elementarily equivalent nonisomorphic finitely generated groups also exist for groups which are much nearer to the abelian class. One of them was given as early as 1971 by B.I. Zil'ber in [18].

It quickly appeared that the class of polycyclic-by-finite groups would be an appropriate setting for an algebraic characterization of elementary equivalence. Actually, two elementarily equivalent such groups necessarily have the same finite images (see [9, Remark, p. 475]) and, by [4], any class of such groups which have the same finite images is a finite union of isomorphism classes.