Journal of Algebra 417 (2014) 1–14

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

# On the existence of cluster tilting objects in triangulated categories

Petter Andreas Bergh

*Department of Mathematical Sciences, NTNU, NO-7491 Trondheim, Norway*

A R T I C L E   I N F O

*Article history:*
Received 27 September 2013
Available online 15 July 2014
Communicated by Michel Van den Bergh

*MSC:*
13D02
13H10
16E05
16E30
18E30

*Keywords:*
Cluster tilting objects
Complexity
Local Gorenstein algebras

A B S T R A C T

We show that in a triangulated category, the existence of a cluster tilting object often implies that the homomorphism groups are bounded in size. This holds for the stable module category of a selfinjective algebra, and as a corollary we recover a theorem of Erdmann and Holm. We then apply our result to Calabi–Yau triangulated categories, in particular stable categories of maximal Cohen–Macaulay modules over commutative local complete Gorenstein algebras with isolated singularities. We show that the existence of almost all kinds of cluster tilting objects can only occur if the algebra is a hypersurface.



## 1. Introduction

Cluster categories of finite dimensional hereditary algebras were introduced in [8] and [11], the latter treating the $A_n$ case. These categories are defined as certain orbit categories of the derived categories of modules. A result of Keller (cf. [20]) shows that they are triangulated, and moreover they are 2-Calabi–Yau. They were introduced as

*E-mail address:* bergh@math.ntnu.no.

http://dx.doi.org/10.1016/j.jalgebra.2014.06.024

a representation theoretic categorification of the combinatorics of the cluster algebras introduced by Fomin and Zelevinsky in [15].

Cluster tilting objects were introduced in [8,18,21] in order to generalize the classical tilting theory for hereditary algebras. In a cluster category, such objects always exist, for example the stalk complex formed by the underlying hereditary algebra. However, the notion of a cluster tilting object makes sense for *any* triangulated category, and it is therefore natural to ask the following:

**Question.** Which triangulated categories contain cluster tilting objects?

We show in this paper that the existence of a cluster tilting object often implies that there is a bound on the size of the homomorphism groups between objects in the category. In Section 2, we prove that this applies to stable module categories of selfinjective algebras. As a corollary, we recover Erdmann and Holm's result [14, Theorem 1.1]: if a cluster tilting object exists for such an algebra, then the minimal projective resolution of every module is bounded. In Section 3, we focus on Calabi–Yau categories, in particular stable categories of maximal Cohen–Macaulay modules over commutative local complete Gorenstein algebras with isolated singularities. We show that the existence of almost all kinds of cluster tilting objects can only occur if the algebra is a hypersurface.

## 2. Cluster tilting and Serre duality

We start by defining cluster tilting objects, or, more precisely, $n$-cluster tilting objects.

**Definition.** Let $(\mathbf{T},\Sigma)$ be a triangulated category and $n$ a positive integer. An object $M$ in $\mathbf{T}$ is *$n$-cluster tilting* if

$$\begin{aligned}\operatorname{add}(M) &= \bigl\{N\in\mathbf{T}\bigm|\operatorname{Hom}_{\mathbf{T}}\bigl(M,\Sigma^iN\bigr)=0 \text{ for } 1\leqslant i\leqslant n-1\bigr\}\\ &= \bigl\{N\in\mathbf{T}\bigm|\operatorname{Hom}_{\mathbf{T}}\bigl(N,\Sigma^iM\bigr)=0 \text{ for } 1\leqslant i\leqslant n-1\bigr\}.\end{aligned}$$

When $n=1$, the vanishing conditions in the definition are empty, and they are trivially satisfied by all the objects in the category. Therefore, a 1-cluster tilting object is an object $M$ with $\operatorname{add}(M)=\mathbf{T}$. If $\mathbf{T}$ is a Krull–Schmidt category, which will be the case in our applications, then such a 1-cluster tilting object exists if and only if the category contains only finitely many isomorphism classes of indecomposable objects.

Suppose now that $k$ is a field and $(\mathbf{T},\Sigma)$ a triangulated Hom-finite $k$-category. Thus $\operatorname{Hom}_{\mathbf{T}}(M,N)$ is a finite dimensional $k$-vector space for all objects $M$, $N$, and composition of morphisms in $\mathbf{T}$ is $k$-bilinear. A *Serre functor* on $\mathbf{T}$ is an equivalence $S\colon\mathbf{T}\to\mathbf{T}$ of triangulated $k$-categories, together with functorial isomorphisms

$$\operatorname{Hom}_{\mathbf{T}}(M,N)\simeq D\operatorname{Hom}_{\mathbf{T}}(N,SM)$$