



Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra



Additive invariants of toric and twisted projective homogeneous varieties via noncommutative motives



Gonalo Tabuada^{a,b,*,1}

^a Department of Mathematics, MIT, Cambridge, MA 02139, USA

^b Departamento de Matemática e CMA, FCT-UNL, Quinta da Torre, 2829-516, Caparica, Portugal

ARTICLE INFO

Article history:

Received 3 June 2014

Available online 15 July 2014

Communicated by Luchezar L. Avramov

Dedicated to the memory of Daniel Kan

MSC:

11E81

14A22

14L17

14M25

18F25

19D55

Keywords:

Homogeneous varieties

Toric varieties

Twisted forms

Torsors

Noncommutative motives

Algebraic K -theory

ABSTRACT

I. Panin proved in the nineties that the algebraic K -theory of twisted projective homogeneous varieties can be expressed in terms of central simple algebras. Later, Merkurjev and Panin described the algebraic K -theory of toric varieties as a direct summand of the algebraic K -theory of separable algebras. In this article, making use of the recent theory of noncommutative motives, we extend Panin and Merkurjev–Panin’s computations from algebraic K -theory to every additive invariant. As a first application, we fully compute the cyclic homology (and all its variants) of twisted projective homogeneous varieties. As a second application, we show that the noncommutative motive of a twisted projective homogeneous variety is trivial if and only if the Brauer classes of the associated central simple algebras are trivial. Along the way we construct a fully-faithful \otimes -functor from Merkurjev–Panin’s motivic category to Kontsevich’s category of noncommutative Chow motives, which is of independent interest.

© 2014 Elsevier Inc. All rights reserved.

* Correspondence to: Department of Mathematics, MIT, Cambridge, MA 02139, USA.

E-mail address: tabuada@math.mit.edu.

URL: <http://math.mit.edu/~tabuada/>.

¹ The author was partially supported by the NSF CAREER Award #1350472 and by the Portuguese Foundation for Science and Technology grant PEst-OE/MAT/UI0297/2014.

1. Introduction

1.1. Algebraic K -theory of twisted projective homogeneous varieties

Let G be a split semisimple algebraic group over a field k , $P \subset G$ a parabolic subgroup, and $\gamma : \mathfrak{g} := \mathrm{Gal}(k_{\mathrm{sep}}/k) \rightarrow G(k_{\mathrm{sep}})$ a 1-cocycle. Out of this data one can construct the projective homogeneous variety $\mathcal{F} := G/P$ as well as its twisted form ${}_{\gamma}\mathcal{F}$. Let \tilde{G} and \tilde{P} be the universal covers of G and P , $R(\tilde{G})$ and $R(\tilde{P})$ the associated representation rings, n the index $[W(\tilde{G}) : W(\tilde{P})]$ of the Weyl groups, \tilde{Z} the center of \tilde{G} , and finally $\mathrm{Ch} := \mathrm{Hom}(\tilde{Z}, \mathbb{G}_m)$ the character group. Under these notations, Panin proved in [21, Thm. 4.2] that every Ch -homogeneous basis ρ_1, \dots, ρ_n of $R(\tilde{P})$ over $R(\tilde{G})$ gives rise to an isomorphism

$$\bigoplus_{i=1}^n K_*(A_{\chi(i), \gamma}) \xrightarrow{\sim} K_*({}_{\gamma}\mathcal{F}), \quad (1.1)$$

where $A_{\chi(i), \gamma}$ stands for the Tits' central simple algebra associated to ρ_i . Panin's computation (1.1) is a landmark in algebraic K -theory. It generalizes previous results of Grothendieck [2] on flag varieties, of Quillen [17] on Severi–Brauer varieties, of Swan [19] and Kapranov [6] on quadrics hypersurfaces, and of Levine, Srinivas and Weyman [15] on twisted Grassmann varieties.

1.2. Algebraic K -theory of toric varieties

Let S be a reductive algebraic group over k and A the associated division separable algebra $\prod_{\rho} A_{\rho} := \prod_{\rho} \mathrm{End}_S(W_{\rho})$, where the product is taken over all irreducible representations $\rho : S \rightarrow \mathrm{GL}(W_{\rho})$. Given an S -torsor $\pi : U \rightarrow X$ over a smooth projective variety X , assume that there exists an S -equivariant open imbedding of U into an affine space on which S acts linearly. Under these assumptions, Merkurjev and Panin proved in [18, Thm. 4.2] that $K_*(X)$ is a direct summand of $K_*(A)$. Examples include toric models (see Example 3.16) and more generally toric varieties (see Remark 7.4).

1.3. Additive invariants

A *dg category* \mathcal{A} , over a field k , is a category enriched over complexes of k -vector spaces; see Section 4. Let \mathbf{dgcat} be the category of (small) dg categories. Every (dg) algebra A gives naturally rise to a dg category with a single object. Another source of examples is provided by schemes since the derived category of perfect complexes $\mathrm{perf}(X)$

Download English Version:

<https://daneshyari.com/en/article/4584654>

Download Persian Version:

<https://daneshyari.com/article/4584654>

[Daneshyari.com](https://daneshyari.com)