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Lattice vertex algebras over fields of prime characteristic

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ABSTRACT

Based on a work of Dong and Griess, we study a class of vertex algebras over fields of prime characteristic obtained from integral forms in lattice vertex operator algebras over complex field. In particular, the generating set is given, a bilinear form is obtained, and the simplicity is studied. Finally, the vertex operator algebra structure is characterized.

1. Introduction

Dong and Griess introduced an integral form of vertex operator algebras (VOA) associated to even lattices (cf. [5]) in [2]. An integral form of a VOA could be considered as an analogue of the integral structure spanned by a Chevalley basis in a Lie algebra or of an integral structure in an enveloping algebra. An integral form of a VOA allows one to create vertex algebras over any field. In this paper, we study vertex algebras over fields of prime characteristic obtained from integral form of lattice VOAs defined by Dong and Griess.

This paper is organized as follows: In Section 2, we present some notations and basic formulas of lattice vertex algebras. Section 3 is divided into four parts. First we give a generating set of lattice vertex algebras over fields of prime characteristic in Subsection 3.1.

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Then we obtain a bilinear form in Subsection 3.2. In Subsection 3.3, we study the simplicity of lattice vertex algebras. Finally we characterize the vertex operator algebra structure in Subsection 3.4.

2. Preliminaries

We will follow the settings of [6] and [2] for the lattice vertex operator algebra V_L and the Heisenberg vertex operator subalgebra $M(1)$ of V_L. Let L be a positive definite even lattice with a basis $\{\gamma_1, \ldots, \gamma_d\}$ and L^* the dual lattice of L. For any $\alpha \in L^*$, set

$$E^-(-\alpha, z) = \exp\left(\sum_{n>0} \frac{\alpha(-n)}{n} z^n\right) := \sum_{n\geq 0} s_{\alpha,n} z^n. \tag{1}$$

Following [2], we denote by $M(1)_{\mathbb{Z}}$ the $\mathbb{Z}$-span of $s_{\alpha_1,n_1} \ldots s_{\alpha_k,n_k}$ for $\alpha_i \in L$ and $n_i \geq 0$, and denote by $(V_L)_{\mathbb{Z}}$ the $\mathbb{Z}$-span of $s_{\alpha_1,n_1} \ldots s_{\alpha_k,n_k} \iota(e^\alpha)$ for $\alpha, \alpha_i \in L$ and $n_i \geq 0$. It was shown in [2] that $M(1)_{\mathbb{Z}}$ is the $\mathbb{Z}$-span of $s_{\alpha_1,n_1} \ldots s_{\alpha_k,n_k}$ for $\alpha_i \in \{\gamma_1, \ldots, \gamma_d\}$ and $n_i \geq 0$, and $(V_L)_{\mathbb{Z}}$ is the $\mathbb{Z}$-span of $s_{\alpha_1,n_1} \ldots s_{\alpha_k,n_k} \iota(e^\alpha)$ for $\alpha \in L$, $\alpha_i \in \{\gamma_1, \ldots, \gamma_d\}$ and $n_i \geq 0$. Moreover $M(1)_{\mathbb{Z}}$ and $(V_L)_{\mathbb{Z}}$ are vertex algebras [1] over $\mathbb{Z}$. Let $\mathbb{F}$ be a field of characteristic $p > 0$. Then $\mathbb{F} \otimes_{\mathbb{Z}} M(1)_{\mathbb{Z}}$ and $\mathbb{F} \otimes_{\mathbb{Z}} (V_L)_{\mathbb{Z}}$ are vertex algebras over $\mathbb{F}$. For simplicity, we denote vertex algebras $\mathbb{F} \otimes_{\mathbb{Z}} M(1)_{\mathbb{Z}}$ and $\mathbb{F} \otimes_{\mathbb{Z}} (V_L)_{\mathbb{Z}}$ by $M(1)_{\mathbb{F}}$ and $V_{L,\mathbb{F}}$, respectively.

In order to characterize the vertex algebra structures of $M(1)_{\mathbb{F}}$ and $V_{L,\mathbb{F}}$, for any $\alpha \in L^*$, we set

$$E^+(-\alpha, z) = \exp\left(\sum_{n>0} \frac{-\alpha(n)}{n} z^{-n}\right) := \sum_{n\geq 0} r_{\alpha,n} z^{-n}. \tag{2}$$

Denote by A_L the $d \times d$ matrix $(\langle \gamma_i, \gamma_j \rangle)$.

For $\alpha, \beta \in L^*$, using formula (cf. [6, Proposition 6.3.14])

$$E^+(-\alpha, z_1) E^-(-\beta, z_2) = \left(1 - \frac{z_2}{z_1}\right)^{\langle \alpha, \beta \rangle} E^-(-\beta, z_2) E^+(-\alpha, z_1), \tag{3}$$

we have

$$r_{\alpha,n} s_{\beta,m} = \sum_{i\geq 0} (-1)^i \binom{\langle \alpha, \beta \rangle}{i} s_{\beta,m-i} r_{\alpha,n-i}, \tag{4}$$

as operators on V_L.

Note that

$$r_{\alpha,0} = 1 = s_{\alpha,0},$$

for $\alpha \in L^*$.