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Universal deformation rings and fusion

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A B S T R A C T

We study in this paper the extent to which one can detect fusion in certain finite groups Γ from information about the universal deformation rings $R(\Gamma, V)$ of absolutely irreducible $\mathbb{F}_p\Gamma$-modules V. The Γ we consider are extensions of either abelian or dihedral groups G of order prime to p by an elementary abelian p-group of rank 2.

1. Introduction

This paper has to do with determining information about the internal structure of a finite group Γ from the knowledge of the universal deformation rings $R(\Gamma, V)$ associated to absolutely irreducible $\mathbb{F}_p\Gamma$-modules V. The kind of internal structure we will consider is the fusion of certain subgroups N in Γ. A pair of elements of N are said to be fused in Γ if they are conjugate in Γ, but not in N. By determining the fusion of N in Γ, we mean listing all such pairs. The universal deformation ring $R(\Gamma, V)$ is characterized by the property that the isomorphism class of every lift of V over a complete local commutative Noetherian ring R with residue field $\mathbb{F}_p$ arises from a unique local ring homomorphism $\alpha : R(\Gamma, V) \to R$. Our main goal is thus to determine how to transfer

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information about the universal deformation rings to information about the structure of groups. It is natural to expect a connection with fusion because fusion plays a key role in the character theory of Γ, which in turn enters into finding universal deformation rings of representations.

In this paper, we consider Γ which are extensions of a group G whose order is relatively prime to p by an elementary abelian p-group N of rank 2. We can now state our main result:

Theorem 1.1. *Let G be a dihedral group of order $2n \geq 6$ and let p be an odd prime such that $p \equiv 1 \bmod n$. Fix an irreducible action of G on $N = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, and let Γ be the resulting semi-direct product of G with N.*

a. If the center of G acts trivially on N, then one can determine the fusion of N in Γ from the absolutely irreducible $\mathbb{F}_p\Gamma$-modules V of dimension 2 over $\mathbb{F}_p$ which have universal deformation ring $R(\Gamma, V)$ different from $\mathbb{Z}_p$.

b. If the center of G acts non-trivially on N, then n is even and $R(\Gamma, V) \cong \mathbb{Z}_p$ for all absolutely irreducible $\mathbb{F}_p\Gamma$-modules V of dimension 2 over $\mathbb{F}_p$. In this case, one can determine the fusion of N in Γ if and only if n is either a power of 2, or $n = 2q$, for some odd prime q.

In Section 4.6 we prove a weaker result when G is abelian. In the course of proving Theorem 1.1, we must calculate $\mathrm{H}^i(\Gamma, \mathrm{Hom}_{\mathbb{F}_p}(V, V))$, for $i = 1, 2$, since these enter into the computation of $R(\Gamma, V)$.

The paper is organized as follows. In Section 2, we recall the definitions of deformations and deformation rings, including some basic results. In Section 3, we concentrate on the case when Γ is an extension of a finite group G by an elementary abelian p-group of rank $\ell \geq 2$. We give an explicit formula for the cohomology groups $\mathrm{H}^i(\Gamma, \mathrm{Hom}_{\mathbb{F}_p}(V, V))$ for $i = 1, 2$ for all projective $\mathbb{F}_pG$-modules V which are viewed as $\mathbb{F}_p\Gamma$-modules by inflation (see Theorem 3.1). In Section 4, we prove our main results, Theorems 4.2 and 4.3, on the connection between fusion and universal deformation rings, respectively cohomology groups, in the case when G is a dihedral group. In Section 4.6, we briefly discuss the case when G is abelian and compare this case to the dihedral one.

This paper is part of my dissertation at the University of Iowa under the supervision of Professor Frauke Bleher [5]. I would like to thank her for all of her advice and guidance.

2. Preliminaries

In this section, we give a brief introduction to universal deformation rings and deformations. For more background material, we refer the reader to [4] and [3].

Let p be an odd prime, $\mathbb{F}_p$ be the field with p elements, and $\mathbb{Z}_p$ denote the ring of p-adic integers. Let $\hat{\mathcal{C}}$ be the category of all complete local commutative Noetherian rings with residue field $\mathbb{F}_p$. Note that all rings in $\hat{\mathcal{C}}$ have a natural $\mathbb{Z}_p$-algebra structure. The