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The group fixed by a family of endomorphisms of a surface group

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ABSTRACT

For a closed surface S with $\chi(S) < 0$, we show that the fixed subgroup of a family $\mathcal{B}$ of endomorphisms of $\pi_1(S)$ has $\operatorname{rank} \operatorname{Fix} \mathcal{B} \le \operatorname{rank} \pi_1(S)$. In particular, if $\mathcal{B}$ contains a non-epimorphic endomorphism, then $\operatorname{rank} \operatorname{Fix} \mathcal{B} \le \frac{1}{2} \operatorname{rank} \pi_1(S)$. We also show that geometric subgroups of $\pi_1(S)$ are inert, and hence the fixed subgroup of a family of epimorphisms of $\pi_1(S)$ is also inert.

1. Introduction

For a finitely generated group G, we denote the rank (i.e., the minimal number of the generators) of G by $\operatorname{rank} G$. There are lots of research on the intersection of subgroups of a finitely generated group G in the literature. For example, when G is a free group,

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H. Neumann (see [18] and [19]) conjectured that for any two finitely generated subgroups A and B of G,

$$\operatorname{rank}(A \cap B) - 1 \leq (\operatorname{rank} A - 1)(\operatorname{rank} B - 1).$$

This conjecture was proved independently by I. Mineyev [17] and J. Friedman [7].

Before this celebrated result was proved, it had been shown that for some special subgroups of free groups, one could say more about their intersection. Denote the set of endomorphisms of G by $\operatorname{End}(G)$. For a family $\mathcal{B}$ of endomorphisms of G, namely, $\mathcal{B} \subseteq \operatorname{End}(G)$, the subgroup fixed by $\mathcal{B}$ is

$$\operatorname{Fix}(\mathcal{B}) := \{ g \in G \mid \phi(g) = g, \forall \phi \in \mathcal{B} \}.$$

It is called the *fixed subgroup* of $\mathcal{B}$. We abbreviate $\operatorname{Fix}(\mathcal{B})$ to $\operatorname{Fix} \mathcal{B}$, and $\operatorname{Fix}(\{\phi\})$ to $\operatorname{Fix} \phi$ for any single endomorphism $\phi : G \to G$ in the context. It is obvious that $\operatorname{Fix} \mathcal{B} = \bigcap_{\phi \in \mathcal{B}} \operatorname{Fix} \phi$.

In [2], M. Bestvina and M. Handel proved Scott's conjecture that for any automorphism ϕ of a finitely generated free group G,

$$\operatorname{rank} \operatorname{Fix} \phi \leq \operatorname{rank} G.$$

In the book [6], W. Dicks and E. Ventura generalized the Bestvina–Handel result on the fixed subgroup of a single automorphism to a family of injective endomorphisms. They proved the following theorem.

Theorem 1.1. *(See [6, Corollary IV.5.8].) Let G be a finitely generated free group, and $\mathcal{B}$ a family of injective endomorphisms of G. Then*

$$\operatorname{rank} \operatorname{Fix} \mathcal{B} \leq \operatorname{rank} G.$$

They also showed that $\operatorname{Fix} \mathcal{B}$ is inert in G (see [6, Theorem IV.5.7]). A subgroup A is *inert* in G if for every subgroup $B \leq G$,

$$\operatorname{rank}(A \cap B) \leq \operatorname{rank} B.$$

In the paper [1], G. Bergman proved that the Dicks–Ventura result (Theorem 1.1) also holds for any family of endomorphisms but kept the following question open: Is the fixed subgroup of a family of endomorphisms of a finitely generated free group inert?

When G is a *surface group*, namely, G is isomorphic to the fundamental group of some connected closed surface S with Euler characteristic $\chi(S) < 0$, T. Soma estimated the rank of the intersection of any two subgroups A and B of G in terms of ranks of A and B. In [22], he showed the following enhanced version of the result of [21]: