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An improved algorithm for deciding semi-definite polynomials

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ABSTRACT

In this paper, a new algorithm is presented for deciding the semi-definiteness of multivariate polynomials with coefficients in a computable ordered field, which admits an effective method of finding an isolating set for every non-zero univariate polynomial. This algorithm is an improvement of the method presented in Ref. [24]. The technique in this paper is to compute triangular decompositions of polynomial systems into regular chains.

1. Introduction

In the computational real algebraic geometry, the decision of semi-definite polynomials is an important topic, which is closely related to many areas, e.g. polynomial optimization, automated theorem proving in ordered geometry, control theory and the study of inequalities.

Let $(K, \leq)$ be a computable ordered field with real closure R, and $K[x_1, \dots, x_n]$ the ring of polynomials in n variables over K. For a non-zero $f(x_1, \dots, x_n) \in K[x_1, \dots, x_n]$,

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we say that f is positive (respectively negative) semi-definite on R if $f(a_1, \ldots, a_n) \geq 0$ (respectively $f(a_1, \ldots, a_n) \leq 0$) for any $a_1, \ldots, a_n \in R$. The decision of semi-definite polynomials is just to devise an algorithm for deciding whether or not a given polynomial is positive semi-definite.

The decision of semi-definite polynomials has been studied extensively by many researchers (for example, see [3,6,7,14]). In Ref. [24], based on the well-known Wu's Algorithm of computing the triangular decompositions of polynomial systems, an effective method was presented for deciding the semi-definiteness of multivariate polynomials with coefficients in a computable ordered field, if this field admits an effective method of finding isolating points for every non-zero univariate polynomial. By this method, the decision of the semi-definiteness of a multivariate polynomial may be reduced to testing some resulted polynomials in fewer variables, of which the total degrees and the term numbers do not exceed those of the given polynomial.

For an input polynomial $f(x_1, \ldots, x_n)$ in n variables, the key of the algorithm in [24] is to compute the triangular decompositions of such polynomial sets $\{f + t, \frac{\partial f}{\partial x_{j_1}}, \ldots, \frac{\partial f}{\partial x_{j_k}}\}$ into irreducible ascending chains, where t is a new variable and $\{j_1, \ldots, j_k\}$ is taken over all the nonempty subsets of $\{1, \ldots, n\}$, see the description of the algorithm in §4 of [24]. The efficiency of this algorithm is thereby dependent on computing the irreducible ascending chains from these polynomial sets. In order to raise the efficiency, one attempt is to remove the polynomial $f + t$ from the involved polynomial sets.

In this paper, we present a new algorithm for deciding the semi-definiteness of multivariate polynomials with coefficients in a computable ordered field, which admits an effective method of finding an isolating set for every univariate polynomial. In this new algorithm, it is enough to compute the weaker triangular sets, the so-called regular chains, instead of irreducible ascending chains, and the involved polynomial sets don't contain the polynomial $f + t$.

Throughout this paper, the symbol K stands for a computable ordered field with real closed extension R. Hence, K and its extensions are fields of characteristic 0. For α, $\beta \in R$ with $\alpha < \beta$, write $]\alpha, \beta[_R$ (or $[\alpha, \beta[_R$) for the open interval $\{z \in R \mid \alpha < z < \beta\}$ (or the closed interval $\{z \in R \mid \alpha \leq z \leq \beta\}$). For a subset P of the polynomial ring $F[x_1, \ldots, x_n]$ over any field F in n variables, denote by (P) the ideal generated by P in $F[x_1, \ldots, x_n]$. Moreover, for a finite set (or sequence) S, ${}^{\#}S$ stands for the number of members in S.

2. Triangular decompositions of polynomial systems into regular chains

In this section, as some preliminaries, we recall some basic concepts and results on the triangular decompositions of polynomial systems, especially on the triangular decompositions of polynomial systems into regular chains.

The triangular decomposition of a polynomial system was introduced by Ritt in [18]. Ritt's decomposition relies on computing the so-called characteristic sets, which are some triangular sets of polynomials, of prime ideals, see [19]. So Ritt's decomposition