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Markov complexity of monomial curves

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ABSTRACT

Let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^m$. We give an algebraic characterization of the universal Markov basis of the toric ideal $I_\mathcal{A}$. We show that the Markov complexity of $\mathcal{A} = \{n_1, n_2, n_3\}$ is equal to 2 if $I_\mathcal{A}$ is complete intersection and equal to 3 otherwise, answering a question posed by Santos and Sturmfels. We prove that for any $r \geq 2$ there is a unique minimal Markov basis of $\mathcal{A}^{(r)}$. Moreover, we prove that for any integer l there exist integers n_1, n_2, n_3 such that the Graver complexity of $\mathcal{A}$ is greater than l.

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Introduction

Let $\Bbbk$ be a field, $n, m \in \mathbb{N}$, $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\} \subset \mathbb{N}^m$ and $A \in \mathcal{M}_{m\times n}(\mathbb{N})$ be the matrix whose columns are the vectors of $\mathcal{A}$. We let $\mathcal{L}(\mathcal{A}) := \mathrm{Ker}_{\mathbb{Z}}(A)$ be the corresponding sublattice of $\mathbb{Z}^n$ and denote by $I_{\mathcal{A}}$ the corresponding toric ideal of $\mathcal{A}$ in $\Bbbk[x_1, \dots, x_n]$. We recall that $I_{\mathcal{A}}$ is generated by all binomials of the form $x^{\mathbf{u}} - x^{\mathbf{w}}$ where $\mathbf{u} - \mathbf{w} \in \mathcal{L}(\mathcal{A})$.

A *Markov basis* of $\mathcal{A}$ is a finite subset $\mathcal{M}$ of $\mathcal{L}(\mathcal{A})$ such that whenever $\mathbf{w}, \mathbf{u} \in \mathbb{N}^n$ and $\mathbf{w} - \mathbf{u} \in \mathcal{L}(\mathcal{A})$ (i.e. $A\mathbf{w} = A\mathbf{u}$), there exists a subset $\{\mathbf{v}_i\colon\ i = 1, \dots, s\}$ of $\mathcal{M}$ that *connects* $\mathbf{w}$ to $\mathbf{u}$. This means that $(\mathbf{w} - \sum_{i=1}^{p} \mathbf{v}_i) \in \mathbb{N}^n$ for all $1 \le p \le s$ and $\mathbf{w} - \mathbf{u} = \sum_{i=1}^{s} \mathbf{v}_i$. A Markov basis $\mathcal{M}$ of $\mathcal{A}$ is *minimal* if no subset of $\mathcal{M}$ is a Markov basis of $\mathcal{A}$. For a vector $\mathbf{u} \in \mathcal{L}(\mathcal{A})$ we let $\mathbf{u}^+$, $\mathbf{u}^-$ be the unique vectors in $\mathbb{N}^n$ such that $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$. If $\mathcal{M}$ is a minimal Markov basis of $\mathcal{A}$ then a classical result of Diaconis and Sturmfels states that the set $\{x^{\mathbf{u}^+} - x^{\mathbf{u}^-}\colon\ \mathbf{u} \in \mathcal{M}\}$ is a minimal generating set of $I_{\mathcal{A}}$, see [6, Theorem 3.1]. The *universal Markov basis* of $\mathcal{A}$, which we denote by $\mathcal{M}(\mathcal{A})$, is the union of all minimal Markov bases of $\mathcal{A}$, where we identify elements that differ by a sign, see [9, Definition 3.1]. The intersection of all minimal Markov bases of $\mathcal{A}$ via the same identification, is called the *indispensable subset* of the universal Markov basis $\mathcal{M}(\mathcal{A})$ and is denoted by $\mathcal{S}(\mathcal{A})$. The *Graver basis* of $\mathcal{A}$, $\mathcal{G}(\mathcal{A})$, is the subset of $\mathcal{L}(\mathcal{A})$ whose elements have no *proper conformal decomposition*, i.e. $\mathbf{u} \in \mathcal{L}(\mathcal{A})$ is in $\mathcal{G}(\mathcal{A})$ if there is no other $\mathbf{v} \in \mathcal{L}(\mathcal{A})$ such that $\mathbf{v}^+ \le \mathbf{u}^+$ and $\mathbf{v}^- \le \mathbf{u}^-$, see [14, Section 4]. The *Graver basis* of $\mathcal{A}$ is always a finite set and contains the universal Markov basis of $\mathcal{A}$, see [14, Section 7]. Thus the following inclusions hold:

$$\mathcal{S}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A}) \subseteq \mathcal{G}(\mathcal{A}).$$

In [4] a description was given for the elements of $\mathcal{S}(\mathcal{A})$ and $\mathcal{M}(\mathcal{A})$ that had a geometrical flavor: it considered the various *fibers* of $\mathcal{A}$ in $\mathbb{N}^n$ and the connected components of certain graphs. It did not examine the problem from a strict algebraic point of view such as conformality. This point of view is seen in [9], but only for the elements of $\mathcal{S}(\mathcal{A})$ from the side of sufficiency. In [9], the authors show that any vector with no *proper semiconformal decomposition* is necessarily in $\mathcal{S}(\mathcal{A})$, see [9, Lemma 3.10]. In this paper we attempt to give the complete algebraic characterization for the elements of $\mathcal{S}(\mathcal{A})$ and $\mathcal{M}(\mathcal{A})$. This is done in Section 1. In Proposition 1.1 we prove that the condition of [9, Lemma 3.10] is not only sufficient but also necessary. We want to point out that the original definition of $\mathcal{S}(\mathcal{A})$ (see [9, Definition 3.9]) is different than ours, but via Proposition 1.1 the two definitions become equivalent.

Next, to give the algebraic characterization of the vectors in $\mathcal{M}(\mathcal{A})$, we introduce the notion of a *proper strongly semiconformal decomposition* and prove that the nonzero vectors with no proper strongly semiconformal decomposition are precisely the vectors of $\mathcal{M}(\mathcal{A})$, see Proposition 1.4. The relationship between these decompositions is given in Lemma 1.2. Schematically the following implications hold:

proper conformal $\Rightarrow$ proper strongly semiconformal $\Rightarrow$ proper semiconformal.