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Reconstructing projective modules from its trace ideal

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ABSTRACT

We make a detailed study of idempotent ideals that are traces of countably generated projective right modules. We associate to such ideals an ascending chain of finitely generated left ideals and, dually, a descending chain of cofinitely generated right ideals.

The study of the first sequence allows us to characterize trace ideals of projective modules and to show that projective modules can always be lifted modulo the trace ideal of a projective module. As a consequence we give some new classification results of (countably generated) projective modules over particular classes of semilocal rings. The study of the second sequence leads us to consider projective modules over noetherian FCR-algebras; we make some constructions of non-trivial projective modules showing that over such rings the behavior of countably generated projective modules that are not direct sum of finitely generated ones is, in general, quite complex.

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It was proved by Whitehead [22] that an idempotent ideal that is finitely generated as a left ideal is the trace of a countably generated projective right R-module. For example, if R is a left noetherian ring then any idempotent ideal is the trace ideal of a countably generated projective right R-module. Příhoda in [13] developed a machinery to work with countably generated projective modules in the setting of noetherian rings. These tools have been extremely helpful in describing non-finitely generated projective modules over suitable classes of noetherian rings with low Krull dimension and not too many idempotent ideals. See [13] for applications to integral group algebras and to some enveloping algebras of Lie algebras, also in [16,15] generalized Weyl algebras and some examples of lattices are studied. Finally, we mention [8] where the behavior of projective modules over semilocal noetherian rings is completely described.

One of the main ideas in Příhoda's techniques is that projective modules over suitable rings can be reconstructed by determining trace ideals and the finitely generated projective modules modulo such trace ideals. In this paper we extend these results on trace ideals of projective modules to a non-necessarily noetherian setting. However, the not so exciting news are that certainly further ideas will be needed to be able to have such complete classifications of projective modules also for some classes of non-noetherian rings. For example, just to understand completely projective modules over semilocal rings seems to be quite a hard problem.

To explain which are the particular properties of trace ideals of projective modules, let us make some observations on trace ideals of the finitely generated ones. Let P be a finitely generated projective right module over a ring R, and let E be an idempotent $n \times n$ matrix with entries in R such that $P \cong ER^n$. The left ideal J generated by the entries of E is finitely generated and satisfies that $J^2 = J$ and the same happens if we consider K to be the right ideal of R generated by the entries of E. Moreover, if I is the trace of P then $I = JR = RK$.

Let I denote now the trace ideal of a countably generated projective right R-module P. Let E be a column-finite countable idempotent matrix defining P, then considering the left ideals generated by the entries in the first columns of E one constructs an ascending chain of finitely generated left ideals $J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n \subseteq \cdots$ such that $J_{n+1}J_n = J_n$ and $I = \bigcup_{n\geq 1} J_n R = I$. The existence of such chains characterizes the ideals that are traces of countably generated projective right R-modules, cf. Proposition 2.4. In Proposition 2.6, we show that trace ideals of arbitrary projective right R-modules can be characterized in terms of the existence of a direct system of ideals having such ascending sequences of finitely generated left ideals. As a consequence, we prove in Corollary 3.2 that projective modules always lift modulo trace ideals of projective module.