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The number of irreducible Brauer characters in a p-block of a finite group with cyclic hyperfocal subgroup

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ABSTRACT

We show that a p-block b of a finite group G with cyclic hyperfocal subgroup and the inertial index e has exactly e irreducible Brauer characters.

1. Introduction

Let p be a prime and $(\mathcal{K}, \mathcal{O}, k)$ be a sufficiently large p-modular system such that k is algebraically closed. Let G be a finite group and let b be a (p)-block of G over $\mathcal{O}$ with a defect pointed group P_γ and with inertial index $e = |N_G(P_\gamma)/PC_G(P)|$. If P is cyclic, then b has e irreducible Brauer characters by Dade [5]. We will generalize this fact. Let Q be the hyperfocal subgroup of P_γ in the sense of Puig [16] or Puig [17], and let $l(b)$ (resp. $k(b)$) be the number of irreducible Brauer (resp. ordinary) characters in b. Set

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$b_0 = (b_P)^{N_G(P)}$, the Brauer correspondent of b. We obtain the following theorem. The purpose of this paper is to prove it.

Theorem 1. *With the notation above, assume Q is cyclic.*

(i) $l(b) = e = l(b_0)$.
(ii) $k(b) = k(b_0)$.

The above theorem is known to be true when $Q = 1$ by Broué and Puig [3]. We shall use the case P is cyclic and the case $Q = 1$ to prove the theorem. It is also a generalization of Watanabe [20], Corollary 2, in which P is abelian. Let b_P be the block of $C_G(P)$ determined by γ. Then (P, b_P) is a maximal b-Brauer pair and $N_G(P, b_P) = N_G(P_\gamma)$. For any subgroup T of P, we denote by (T, b_T) the b-Brauer pair contained in (P, b_P). By [16], 1.7,

$$\begin{aligned} Q &= \langle [t, x] \mid t \in T \leq P,\ x \in N_G(T, b_T)_{p'} \rangle \\ &= \langle [T, O^p(N_G(T, b_T))] \mid T \leq P \rangle \\ &= \langle [T, O^p(N_G(T, b_T)/C_G(T))] \mid T \leq P \rangle, \end{aligned}$$

where we denote by $G_{p'}$ the set of p-regular elements of G and by $O^p(G)$ the normal subgroup of G generated by $G_{p'}$. We call Q the hyperfocal subgroup of b (with respect to (P, b_P)). Clearly $N_G(P, b_P) \subseteq N_G(Q, b_Q)$ by the uniqueness of b_Q. Also, $Q = 1$ if and only if b is a nilpotent block. Then $l(b) = 1 = e$ by [3].

For $a \in \mathcal{O}G$, we denote by a^* the image of a in kG by the canonical homomorphism. Let i be the image in kG of an element of γ, and set $B = ikGi$. Hence i is a source idempotent of b^*, and B is a source algebra of b^* (Thévenaz [19], §38). By [17], §13–§15 and [16], there is a P-stable interior kQ-algebra D such that

$$D \cap Pi = Qi \quad \text{and} \quad B = \bigoplus_{u \in [Q \backslash P]} Du.$$

We call D a hyperfocal subalgebra of b^* in B. Indeed a hyperfocal subalgebra is defined for b.

Let $\mathcal{F} = \mathcal{F}_{(P, b_P)}(G, b)$ be the Brauer category of b with respect to the maximal b-Brauer pair (P, b_P). The objects of $\mathcal{F}$ are the b-Brauer pairs contained in (P, b_P) and the set $\mathrm{Mor}_{\mathcal{F}}((S, b_S), (T, b_T))$ of morphisms from (S, b_S) to (T, b_T) is the set of all group homomorphisms $\phi : S \to T$ such that there exists $g \in G$ satisfying $(S, b_T)^g \leq (T, b_T)$ and $\phi(s) = s^g$ for all $s \in S$. As is well known, $\mathcal{F}$ is a saturated fusion system when we regard $\mathcal{F}$ as a fusion system over P. We refer Aschbacher et al. [2] and Craven [4] for fusion systems. For $T \leq P$, (T, b_T) is extremal in (P, b_P) in the sense of Alperin and Broué [1] if and only if $N_P(T)$ is a defect group of b_T as a block of $N_G(T, b_T)$.