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Layered tropical mathematics

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ABSTRACT

Generalizing supertropical algebras, we present a "layered" structure, "sorted" by a semiring which permits varying ghost layers, and indicate how it is more amenable than the "standard" supertropical construction in factorizations of polynomials, description of varieties, and for mathematical analysis and calculus, in particular with respect to multiple roots of polynomials. This gives rise to a significantly better understanding of the tropical resultant and discriminant. Explicit examples and comparisons are given for various sorting semirings such as the natural numbers and the positive rational numbers, and we see how this theory relates to some recent developments in the tropical literature.

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Contents

1. Introduction

Tropical geometry, a rapidly growing area expounded for example in [6,7,10,29,30,37], has been based on two main approaches. Primarily, tropical curves have been defined as domains of non-differentiability of polynomials over the ordered group $(R,+)$, viewed as the max-plus algebra, and also in terms of valuation theory applied to curves over Puiseux series. Unfortunately, semirings such as the max-plus algebra possess a limited algebraic structure theory, and also do not reflect the valuation-theoretic properties intrinsic in tropical mathematics, thereby forcing researchers to turn to combinatoric arguments.

To enhance the availability of standard algebraic techniques, the authors of this paper have modified the max-plus construction over the last few years. The first author introduced a modification in [11] of idempotent semirings, which in [20] evolved to the supertropical semiring, which we call here the **standard** supertropical semiring. Its theory is far more compatible with algebraic structure theory and valuation theory than the max-plus algebra, and has been investigated in a sequence of papers including [21,22,24, 15] which concern matrices, and [25], which focuses on various fundamental properties involving roots of polynomials and resultants, as well as [14] and [17] which deal directly with supertropical valuation theory. The basic idea of this theory is to introduce another "ghost" copy $\mathbb{R}^\nu$ of the max-plus algebra $\mathbb{R}_{\max,+}$ (graded by $\{1,\infty\}$, where by definition $1\cdot 1 = 1$ and every other sum and product is ∞), which provides a semiring R that is a cover of the max-plus algebra $\mathbb{R}_{\max,+}$ in which we can "resolve" additive idempotents, in the sense that $a+a=a^\nu$ instead of $a+a=a$.[4] This modification permits us to detect corner roots of polynomials in terms of the algebraic structure by means of ghosts.

[4] One can think of the ghost elements as uncertainties in classical algebra arising from adding two Puiseux series whose lowest order terms have the same degree.