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Left 3-Engel elements in groups of exponent 5

Gunnar Traustason

Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK

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ABSTRACT

It is still an open question whether a left 3-Engel element of a group G is always contained in the Hirsch–Plotkin radical of G. In this paper we begin a systematic study of this problem. The problem is first rephrased as saying that a certain type of groups are locally nilpotent. We refer to these groups as sandwich groups as they can be seen as the analogs of sandwich algebras in the context of Lie algebras. We show that any 3-generator sandwich group is nilpotent and obtain a power-conjugation presentation for the free 3-generator sandwich group. As an application we show that the left 3-Engel elements in any group G of exponent 5 are in the Hirsch–Plotkin radical of G.

1. Introduction

Let G be a group. An element $a \in G$ is a left Engel element in G, if for each $x \in G$ there exists a non-negative integer $n(x)$ such that

$$[[[x, \underbrace{a], a], \ldots, a}_{n(x)}] = 1.$$

If $n(x)$ is bounded above by n then we say that a is a left n-Engel element in G. It is straightforward to see that any element of the Hirsch–Plotkin radical $HP(G)$ of G is a left Engel element and the converse is known to be true for some classes of groups,

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including solvable groups and finite groups (more generally groups satisfying the maximal condition on subgroups) [3,6]. The converse is however not true in general and this is the case even for bounded left Engel elements. In fact whereas one sees readily that a left 2-Engel element is always in the Hirsch–Plotkin radical this is still an open question for left 3-Engel elements. There is some substantial progress by A. Abdollahi in [1] where he proves in particular that for any left 3-Engel p-element a in a group G one has that a^p is in $HP(G)$ (in fact he proves the stronger result that a^p is in the Baer radical), and that the subgroup generated by two left 3-Engel elements is nilpotent of class at most 4. See also [2] for some results about left 4-Engel elements.

Groups of prime power exponent are known to satisfy some Engel type conditions and the solution to the restricted Burnside problem in particular makes use of the fact that the associated Lie ring satisfies certain Engel type identities [12,13]. Considering left Engel elements, it was observed by William Burnside [4] that every element in a group of exponent 3, is a left 2-Engel element and so the fact that every left 2-Engel element lies in the Hirsch–Plotkin radical can be seen as the underlying reason why groups of exponent 3 are locally finite. For groups of 2-power exponent there is a close link with left Engel elements. Let G be a finitely generated group of exponent 2^n and a an element in G of order 2, then

$$[[[x, \underbrace{a], a], \dots, a}_{n+1}] = [x, a]^{(-2)^n} = 1.$$

Thus a is a left $(n + 1)$-Engel element of G. It follows from this that if $G/G^{2^{n-1}}$ is finite and the left $(n + 1)$-Engel elements of G are in the Hirsch–Plotkin radical, then G is finite. As we know that for sufficiently large n the variety of groups of exponent 2^n is not locally finite [8,9], it follows that for sufficiently large n there are left n-Engel elements that are not contained in the Hirsch–Plotkin radical. Notice also that if all left 4-Engel elements of a group G of exponent 8 are in $HP(G)$, then G is locally finite.

In this paper we focus on left 3-Engel elements. We first make the observation that an element $a \in G$ is a left 3-Engel element if and only if $\langle a, a^x \rangle$ is nilpotent of class at most 2 for all $x \in G$ [1]. We next introduce a related class of groups.

Definition. A *sandwich* group is a group G generated by a set X of elements such that $\langle x, y^g \rangle$ is nilpotent of class at most 2 for all $x, y \in X$ and all $g \in G$.

If $a \in G$ is a left 3-Engel element then $H = \langle a \rangle^G$ is a *sandwich* group and it is clear that the following statements are equivalent:

(1) For every pair (G, a) where a is a left 3-Engel element in the group G we have that a is in the locally nilpotent radical of G.
(2) Every sandwich group is locally nilpotent.