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The geometry of arithmetic noncommutative projective lines

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Let k be a perfect field and let K/k be a finite extension of fields. An arithmetic noncommutative projective line is a noncommutative space of the form $\mathsf{Proj}\,\mathbb{S}_K(V)$, where V be a k-central two-sided vector space over K of rank two and $\mathbb{S}_K(V)$ is the noncommutative symmetric algebra generated by V over K defined by M. Van den Bergh [26]. We study the geometry of these spaces. More precisely, we prove they are integral, we classify vector bundles over them, we classify them up to isomorphism, and we classify isomorphisms between them. Using the classification of isomorphisms, we compute the automorphism group of an arithmetic noncommutative projective line.

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1. Introduction

Throughout this paper, k will denote a perfect field and K/k will be a finite extension of fields. The purpose of this paper is to study the geometry of noncommutative spaces (i.e. Grothendieck categories) of the form $\mathsf{Proj}\,\mathbb{S}_K(V)$, where V is a k-central two-sided vector space over K of rank two, i.e. a k-central K–K-bimodule which is two-dimensional as a vector space via the right and left actions of K on V, $\mathbb{S}_K(V)$ is the noncommutative symmetric algebra defined by Van den Bergh [26], and $\mathsf{Proj}\,\mathbb{S}_K(V)$ denotes the quotient of the category of graded right $\mathbb{S}_K(V)$-modules modulo the full subcategory of direct limits of right bounded modules. We denote the noncommutative space by $\mathbb{P}_K(V)$, and we refer to this space as an *arithmetic* noncommutative projective line since, as we shall show, its geometry is intimately connected to data associated with K/k. In the sequel, we shall drop the term "arithmetic". We hope that noncommutative projective lines have some utility as basic examples arising in the relatively new field of noncommutative arithmetic geometry.

Our primary motivation for the study of noncommutative projective lines is Artin's conjecture [1], which states that the division ring of fractions of a noncommutative surface not finite over its center is the function field of a noncommutative $\mathbb{P}^1$-bundle over a smooth commutative curve (see [26] for a definition of the latter space). The investigations in this paper do not directly address this conjecture, but provide what physicists might call a toy model for the geometry of noncommutative $\mathbb{P}^1$-bundles over smooth commutative curves, since a noncommutative projective line is just a noncommutative $\mathbb{P}^1$-bundle over the point $\operatorname{Spec} K$. In Theorem 1.2, stated below, we obtain isomorphism invariants of noncommutative projective lines. It is not yet known whether these are also birational invariants. If Artin's conjecture is true, then noncommutative surfaces infinite over their center will share birational invariants with noncommutative $\mathbb{P}^1$-bundles over smooth commutative curves, and we hope our work suggests the form these invariants take.

In order to justify the name 'noncommutative projective line', these spaces should have geometric properties in common with the commutative projective line, and they do: it is known that they are noetherian [26, Section 6.3] Ext-finite [12, Corollary 3.6]