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Exact couples in semiabelian categories revisited

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ABSTRACT

Consider an exact couple in a semiabelian category in the sense of Palamodov, i.e., in an additive category in which every morphism has a kernel as well as a cokernel and the induced morphism between coimage and image is always monic and epic. Assume that the morphisms in the couple are strict, i.e., they induce even isomorphisms between their corresponding coimages and images. We show that the classical construction of Eckmann and Hilton in this case produces two derived couples which are connected by a natural bimorphism. The two couples correspond to the a priori distinct cohomology objects, the left resp. right cohomology, associated with the initial exact couple. The derivation process can be iterated under additional assumptions.

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1. Preliminaries

The aim of this note is to elaborate rigorously in which way the classical approach of Eckmann, Hilton [1] for constructing exact couples, see Massey [7], generalizes to semiabelian categories. For quasiabelian categories this was carried out by Kopylov [4] in 2004; the negative answer to Raĭkov's conjecture given around 2008, see Rump [8], showed however that the class of semiabelian categories is strictly larger. We point out that semiabelian categories appear in different branches of mathematics, see Kopylov, Wegner [6, Section 1] for more details and references. For recent results on exact couples in non-additive situations we refer to Grandis [2].

In the sequel, $\mathcal{A}$ always denotes a preabelian category, i.e., $\mathcal{A}$ is additive and every morphism α in $\mathcal{A}$ has a kernel and a cokernel. We denote by $\bar{\alpha}\colon \operatorname{Coim}\alpha \to \operatorname{Im}\alpha$ the canonical morphism. The category $\mathcal{A}$ is semiabelian if the induced morphism $\bar{\alpha}$ is always monic and epic, viz., a bimorphism. We say that a morphism α is strict if $\bar{\alpha}$ is an isomorphism. We say that a kernel α is semistable if all its pushouts along arbitrary morphisms are again kernels. Semi-stable cokernels are defined dually. An exact couple in $\mathcal{A}$ is a diagram of the form

$$\begin{array}{ccc} D & \xrightarrow{\quad\alpha\quad} & D \\ & {\scriptstyle\gamma}\nwarrow \qquad \swarrow{\scriptstyle\beta} & \\ & E & \end{array} \tag{1}$$

such that $\operatorname{im}\alpha = \ker\beta$, $\operatorname{im}\beta = \ker\gamma$ and $\operatorname{im}\gamma = \ker\alpha$ hold. If $\mathcal{A}$ is semiabelian, it is easy to see that the latter equations are equivalent to $\operatorname{cok}\alpha = \operatorname{coim}\beta$, $\operatorname{cok}\beta = \operatorname{coim}\gamma$ and $\operatorname{cok}\gamma = \operatorname{coim}\alpha$, respectively. E.g., $\operatorname{coim}\beta = \operatorname{cok}\operatorname{im}\alpha = \operatorname{cok}((\operatorname{im}\alpha)\bar{\alpha}\operatorname{coim}\alpha) = \operatorname{cok}\alpha$ and the dual computation yield the first equivalence.

2. Results

Theorem 1. *Let $\mathcal{A}$ be semiabelian and consider the exact couple (1). Assume that α, β and γ are strict. Then the Eckmann–Hilton construction, see Section 3, gives rise to the following two diagrams*

$$\begin{array}{ccc} D_1 & \xrightarrow{\quad\alpha_1\quad} & D_1 \\ & {\scriptstyle\gamma_1^-}\nwarrow \qquad \swarrow{\scriptstyle\beta_1^-} & \\ & E_1^- & \end{array} \qquad\qquad \begin{array}{ccc} D_1 & \xrightarrow{\quad\alpha_1\quad} & D_1 \\ & {\scriptstyle\gamma_1^+}\nwarrow \qquad \swarrow{\scriptstyle\beta_1^+} & \\ & E_1^+ & \end{array} \tag{2}$$

which we call the left *resp. the* right derived couple*. The diagrams in (2) have the following properties.*