Journal of Algebra 412 (2014) 150–154

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Cocharacter sequences are holonomic

Allan Berele

Department of Mathematics, DePaul University, Chicago, IL 60614, United States

ARTICLE INFO

Article history:
Received 2 September 2013
Available online 20 May 2014
Communicated by Louis Rowen

Keywords:
P.i. algebras
Codimension sequences
Holonomic sequences

ABSTRACT

Let $\{c_n\}_{n=0}^{\infty}$ be a codimension sequence of a characteristic zero p.i. algebra. Then the c_n satisfy a linear, homogenous recurrence relation whose coefficients are rational functions of n.

Let A be a p.i. algebra in characteristic zero with codimension sequence $\{c_n\}_{n=0}^{\infty}$. Our main theorem is that there exist rational functions $q_1(x), \dots, q_m(x)$ such that for all large n,

$$c_n = c_{n-1}q_1(n) + \cdots + c_{n-m}q_m(n). \tag{1}$$

The proof follows easily by combining the theory of holonomic sequences developed in [5], and the application of hook Schur functions to cocharacters from [2]. We are grateful to Doron Zeilberger for suggesting that the theory of holonomic sequences would be helpful for proving (1) and for pointing out [5] to us to learn about them. We happily perform the latter service to the reader and merely cite the theorems we need from [5]. We consider functions $c : \mathbb{N}^k \to \mathbb{R}$.

Holonomic Theorem 1. *If $k = 1$ and c is holonomic, then c satisfies a recurrence relation as in (1).*

http://dx.doi.org/10.1016/j.jalgebra.2014.02.030
0021-8693/© 2014 Published by Elsevier Inc.

Based on this theorem we now focus on proving that the sequence $\{c_n\}$ is holonomic.

Holonomic Theorem 2. *If $c_1, c_2 : \mathbb{N}^k \to \mathbb{R}$ are each holonomic, so is their sum and product.*

Holonomic Theorem 3. *If $c : \mathbb{N}^k \to \mathbb{R}$ is holonomic, so is $c'(n) = \sum_{n_1+\cdots+n_k=n} c(n_1, \ldots, n_k)$.*

Holonomic Theorem 4. *If $\sum c(n_1, \ldots, n_k)x^{n_1} \cdots x^{n_k}$ is the Taylor series of a rational function $P(x_1, \ldots, x_k)/Q(x_1, \ldots, x_k)$ such that $Q(0, \ldots, 0) \neq 0$, then the coefficient function c is holonomic.*

Holonomic Theorem 5. *If $c(n_1, \ldots, n_k)$ is a product or quotient of a polynomial in the n_i and factorial functions $(a_0 + a_1 n_1 + \ldots + a_k n_k)!$, where each $a_i \in \mathbb{N}$, then c is holonomic.*

We now turn to codimensions. Recall that A is a p.i. algebra in characteristic zero with codimension sequence $\{c_n\}_{n=0}^{\infty}$. It is well-known that each codimension can be written as a sum

$$c_n = \sum_{\lambda \vdash n} m_\lambda f^\lambda, \tag{2}$$

where the m_λ are the multiplicities of the irreducible components in the cocharacter sequence and f^λ is the dimension of the irreducible S_n-character corresponding to λ. The next ingredient is the Amitsur–Regev theorem, also proven by Kemer in a different form, see [1] and [4].

PI Theorem 1 *(Amitsur–Regev). Let $H(k, \ell)$, the k by ℓ hook, denote the set of all partitions with at most k parts greater than ℓ. Then there exist $k, \ell \geq 0$ such that $m_\lambda = 0$ unless $\lambda \in H(k, \ell)$.*

The hook Schur functions $HS_\lambda(x_1, \ldots, x_k; y_1, \ldots, y_\ell)$ have the property that they are zero if λ lies outside of $H(k, \ell)$, and are linearly independent otherwise. Hence, one defines the double Poincaré series of A as

$$P_{k,\ell}(A) = \sum m_\lambda HS_\lambda(x_1, \ldots, x_k; y_1, \ldots, y_\ell)$$

and it completely encodes the cocharacter sequence. The following theorem of Berele was proven in [2].

PI Theorem 2. *The double Poincaré series $P_{k,\ell}(A)$ is the Taylor series of a rational function with denominator a product of terms $(1 - u)$ where u is a monic monomial.*