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Irreducible characters of wreath products in reality-based algebras and applications to association schemes

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A B S T R A C T

Wreath products in reality-based algebras are generalizations of wreath products of table algebras and generalized Camina–Frobenius pairs of C-algebras. In this paper we present characterizations of the wreath product in a reality-based algebra by its irreducible characters and by the size of the zero submatrix of its character table. Applications to finite groups, table algebras, and association schemes are also discussed. In particular, we will show that the wreath product of one-class association schemes is characterized by the zeros in its first eigenmatrix.

1. Introduction

The wreath product of association schemes provides a useful way to construct new association schemes from old ones (cf. [10,16], etc.). The wreath product of table algebras was first used by Arad and Muzychuk [3] for the classification of certain classes

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of standard integral table algebras. Since the Bose–Mesner algebra of an association scheme is a standard table algebra, and as a table algebra, the Bose–Mesner algebra of the wreath product of association schemes is exactly isomorphic to the wreath product of the Bose–Mesner algebras of those association schemes, the study of the wreath product of table algebras has natural applications to association schemes. Some basic properties of wreath products of C-algebras, table algebras, and association schemes are presented in [1,15].

Reality-based algebras are generalizations of table algebras. In this paper we study wreath products of standard reality-based algebras. Our main results (Theorems 1.6 and 1.8 below) characterize wreath products of these algebras in terms of their character tables. An immediate consequence of Theorem 1.6 is a result of Arad and Fisman [1, Lemma 2.11] on C-algebras that are wreath products (see Corollary 1.7 below). Applications to finite groups, table algebras, and association schemes are also discussed. In particular, Belonogov's result [7] for the Camina–Frobenius pairs in finite groups (Corollary 1.10 below) is a direct consequence of Theorem 1.8, and the first and second eigenmatrices of the wreath product of one-class association schemes can be easily obtained without calculations from Theorem 1.8 and its proof. Furthermore, by applying Theorem 1.8, we will show that the wreath product of one-class association schemes is characterized by the zeros in its first eigenmatrix.

In the rest of this introductory section, we state the main results of the paper explicitly. Let us start with some necessary definitions and notation.

Definition 1.1. (Cf. [5, Definition 1.16].) A *reality-based algebra* (RBA) $(A, \mathbf{B})$ is a finite dimensional associative algebra A over the complex numbers $\mathbb{C}$ with a distinguished basis $\mathbf{B} := \{b_i \mid 0 \le i \le k\}$, where $b_0 = 1_A$, the identity element of A, and the following three conditions hold.

(i) The structure constants for $\mathbf{B}$ are real numbers; that is, for all $b_i, b_j \in \mathbf{B}$,

$$b_i b_j = \sum_{m=0}^{k} \lambda_{ijm} b_m, \quad \text{for some } \lambda_{ijm} \in \mathbb{R}.$$

(ii) There is an algebra anti-automorphism (denoted by $*$) of A such that $(a^*)^* = a$ for all $a \in A$ and $b_i^* \in \mathbf{B}$ for all $b_i \in \mathbf{B}$. (Hence i^* is defined by $b_{i^*} = b_i^*$.)

(iii) For all $b_i, b_j \in \mathbf{B}$, $\lambda_{ij0} = 0$ if $j \ne i^*$; and $\lambda_{ii^*0} = \lambda_{i^*i0} > 0$.

Let $(A, \mathbf{B})$ be a RBA. If all structure constants λ_{ijm} are nonnegative, then $(A, \mathbf{B})$ is a *table algebra*. A *degree map* (cf. [5, Definition 1.1d]) for $(A, \mathbf{B})$ is an algebra homomorphism $f : A \to \mathbb{C}$ such that $f(b_i) \in \mathbb{R} \setminus \{0\}$ for all $b_i \in \mathbf{B}$. The values $f(b_i)$ are called the *degrees* of $(A, \mathbf{B}, f)$. If $(A, \mathbf{B})$ is a table algebra, then there always exists a (unique) degree map that is positive on $\mathbf{B}$; such a degree map is called the *positive* degree map. If f is a degree map of $(A, \mathbf{B})$ such that $f(b_i) = \lambda_{ii^*0}$ for all i, then $(A, \mathbf{B}, f)$ is called a