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On formations of finite groups with the generalised Wielandt property for residuals

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ABSTRACT

A formation $\mathfrak{F}$ of finite groups has the generalised Wielandt property for residuals, or $\mathfrak{F}$ is a GWP-formation, if the $\mathfrak{F}$-residual of a group generated by two $\mathfrak{F}$-subnormal subgroups is the subgroup generated by their $\mathfrak{F}$-residuals. We prove that every GWP-formation is saturated. This is one of the crucial steps in the hunt for a solution of the classification problem.

1. Introduction

All groups considered in this paper are finite. It is abundantly clear that subnormal subgroups that permute with each other play an important role in the structural study of the groups. Wielandt was concerned with this question and developed an elegant theory

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of operators which allows him to give sufficient conditions to ensure such permutability. In fact, each new operator leads to the discovering of a new case of permutability of subnormal subgroups.

Wielandt's theory of operators is closely related to the theory of formations: the problem of finding Wielandt operators which are permutable with homomorphisms is reduced to the description of the formations $\mathfrak{F}$ satisfying the *Wielandt property for residuals*: if G is a group generated by two subnormal subgroups A and B, then $G^{\mathfrak{F}} = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle$. Here, $G^{\mathfrak{F}}$ is the $\mathfrak{F}$-residual of G, that is, the smallest normal subgroup of G with quotient in $\mathfrak{F}$.

Every formation satisfying the Wielandt property for residuals must be a Fitting class. The validity of the converse is not known at the time of writing and seems to be extremely difficult. However, some interesting partial results are available. They allow us to discover that every soluble subgroup-closed Fitting formation satisfies the Wielandt property for residuals. In particular, the nilpotent residual of a group generated by two subnormal subgroups is generated by the nilpotent residual of the subgroups, and the same is true for the supersoluble and soluble residuals (see [3, Section 6.5] for details).

Bearing in mind the extension of the subnormality within the framework of formation theory and its strong influence on the theory of subnormal subgroups (see [3, Chapter 6]), it seems interesting to study the corresponding extension of Wielandt property to this context in order to help us to better understand the original one.

Let $\mathfrak{F}$ be a formation, a class of groups which is closed under taking epimorphic images and subdirect products. A maximal subgroup M of a group G is said to be $\mathfrak{F}$-normal in G if the primitive group $G/\mathrm{Core}_G(M)$ belongs to $\mathfrak{F}$. It is clear that M is $\mathfrak{F}$-normal in G if and only if M contains $G^{\mathfrak{F}}$.

Definition 1. Let $\mathfrak{F}$ be a formation. A subgroup U of a group G is called an $\mathfrak{F}$-subnormal subgroup of G if either $U = G$ or there is a chain of subgroups

$$U = U_0 < U_1 < \ldots < U_n = G$$

such that U_{i-1} is a maximal $\mathfrak{F}$-normal subgroup of U_i, for $i = 1, 2, \ldots, n$.

It is rather clear that the $\mathfrak{N}$-subnormal subgroups of a group G for the formation $\mathfrak{N}$ of all nilpotent groups are subnormal, and they coincide in the soluble universe.

Definition 2. Let $\mathfrak{F}$ be a non-empty formation. We say that $\mathfrak{F}$ has the *generalised Wielandt property for residuals*, $\mathfrak{F}$ is a *GWP-formation* for short, if $\mathfrak{F}$ enjoys the following property: If G is a group generated by two $\mathfrak{F}$-subnormal subgroups A and B, then $G^{\mathfrak{F}} = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle$.

Although there exist formations satisfying the Wielandt property for residuals which are not GWP-formations [3, 6.5.31], the complete description of the GWP-formations could give us clues about whether or not every Fitting formation has the Wielandt property for residuals. This problem has been extensively studied in the papers [2,8,7,4] and appears as an open question in [3, p. 301]. Unfortunately, there is not a solution at the time of writing despite the fact that they have strong and interesting properties.