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A family of solutions of the Yang–Baxter equation

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ABSTRACT

A new method to construct involutive non-degenerate set-theoretic solutions $(X^n, r^{(n)})$ of the Yang–Baxter equation, for any positive integer n, from a given solution (X, r) is presented. Furthermore, the permutation group $\mathcal{G}(X^n, r^{(n)})$ associated with the solution $(X^n, r^{(n)})$ is isomorphic to a subgroup of $\mathcal{G}(X, r)$, and in many cases $\mathcal{G}(X^n, r^{(n)}) \cong \mathcal{G}(X, r)$.

1. Introduction

The quantum Yang–Baxter equation is one of the basic equations in mathematical physics named after the authors of the first works in which the equation arose: the solution of the delta function Fermi gas by C.N. Yang [15], and the solution of the 8-vertex model by R.J. Baxter [1]. It also lies at the foundations of the theory of quantum groups. One of the important open problems related to this equation is to compute all

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its solutions. Those are linear maps $R : V \otimes V \to V \otimes V$, with V a vector space, that satisfy

$$R_{12} \circ R_{13} \circ R_{23} = R_{23} \circ R_{13} \circ R_{12},$$

where R_{ij} denotes the map $R_{ij} : V \otimes V \otimes V \to V \otimes V \otimes V$ acting as R on the (i,j) tensor factors and as the identity on the remaining factor.

Finding all the solutions of the Yang–Baxter equation is a difficult task far from being solved. Nevertheless, many solutions have been found during the last 20 years and the related algebraic structures (Hopf algebras in particular) have been studied.

In [5], Drinfeld suggested the study of a simpler case: solutions induced by a linear extension of a mapping $\mathcal{R} : X \times X \to X \times X$, where X is a basis for V. In this case, one says that $\mathcal{R}$ is a set-theoretic solution of the quantum Yang–Baxter equation. It is not difficult to see that, if $\tau : X^2 \to X^2$ is the map defined by $\tau(x,y) = (y,x)$, then the map $\mathcal{R} : X^2 \to X^2$ is a set-theoretic solution of the quantum Yang–Baxter equation if and only if the mapping $r = \tau \circ \mathcal{R}$ is a solution of the equation

$$r_{12} \circ r_{23} \circ r_{12} = r_{23} \circ r_{12} \circ r_{23},$$

where r_{ij} is the map from X^3 to X^3 that acts as r on the (i,j) components and as the identity on the remaining component. In the sequel, we will always work with this last equivalent equation.

We study solutions with some additional conditions: involutivity and non-degeneracy. A map

$$\begin{array}{rccc} r: & X \times X & \longrightarrow & X \times X \\ & (x,y) & \longmapsto & \big(\sigma_x(y), \gamma_y(x)\big) \end{array}$$

is said to be involutive if $r \circ r = \mathrm{id}_{X^2}$. Moreover, it is said to be left (resp. right) non-degenerate if each map σ_x (respectively, γ_y) is bijective, and it is said to be non-degenerate if it is left and right non-degenerate. If r is involutive and left non-degenerate, it can be checked that it satisfies $r_{12} \circ r_{23} \circ r_{12} = r_{23} \circ r_{12} \circ r_{23}$ if and only if it satisfies $\sigma_x \circ \sigma_{\sigma_x^{-1}(y)} = \sigma_y \circ \sigma_{\sigma_y^{-1}(x)}$ for all $x, y \in X$ (see [6, Proposition 2.2] or the proof of [12, Theorem 9.3.10]).

In what follows, by a solution of the YBE we will mean a non-degenerate involutive set-theoretic solution of the Yang–Baxter equation.

In the last years, solutions of the YBE have received a lot of attention [3,4,6–11,13,14]. Each solution (X,r) of the YBE has an associated structure group, denoted by $G(X,r)$, and defined by

$$G(X,r) := \big\langle x \in X \mid xy = zt \text{ if and only if } r(x,y) = (z,t) \big\rangle.$$