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Equivariant cohomology and sheaves

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ABSTRACT

RO(G)-graded equivariant cohomology was originally defined by using equivariant spectra. In this paper we construct a sheaf version of RO(G)-graded equivariant cohomology theory. By introducing a complex of presheaves we obtain an isomorphism theorem of two different cohomology theories. We further apply the results to the study of real algebraic varieties.

1. Introduction

For a finite group G, Illman [10] showed that every smooth G-manifold admits a smooth equivariant triangulation onto a regular simplicial G-complex. With this result we extend to the equivariant context a well-known classical theorem [1, p. 42] about the existence of good covers on a smooth manifold.

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Theorem. *(See Theorem 2.11.) Every smooth G-manifold has an equivariant good cover. Moreover, the equivariant good covers are cofinal in the set of all open covers.*

Here an open cover $\mathscr{U} = \{U_\alpha\}_{\alpha \in I}$ of a smooth G-manifold X is called an *equivariant good cover* if it is a regular G-cover (Definition 2.7) and for each subgroup $H \leqslant G$, when restricted to the fixed point set X^H, $\mathscr{U}^H \overset{\text{def}}{=} \{U_\alpha \cap X^H\}_{\alpha \in I}$ is a good cover of X^H.

On the other hand, let $\mathrm{RO}(G)$ be the real representation ring of G. An $\mathrm{RO}(G)$-graded cohomology theory is defined on any G-space X [12,15]. It is a cohomology theory $\{H^V(X, M) \mid V \in \mathrm{RO}(G)\}$ on X graded by the ring $\mathrm{RO}(G)$, with coefficients in a Mackey functor M. We call it $\mathrm{RO}(G)$-graded Bredon cohomology theory, since the ordinary equivariant cohomology theory was first defined by Bredon [3].

One of the concerns about the $\mathrm{RO}(G)$-graded cohomology theory is how it relates to the Čech hypercohomology. For a complex of presheaves $\mathcal{F}^*$ and an open cover $\mathscr{U}$ of a G-space X, let $\check{\mathbb{H}}^n_G(\mathscr{U}, \mathcal{F}^*)$ denote the n-th Čech hypercohomology group with coefficients in $\mathcal{F}^*$. It is obtained by first forming a double complex $\mathscr{C}^{**}$ over $\mathcal{F}^*$, where $\mathscr{C}^{pq} = \mathscr{C}^p(\mathscr{U}, \mathcal{F}^q)$ and $\mathscr{C}^*(\mathscr{U}, \mathcal{F}^q)$ is the Čech resolution of $\mathcal{F}^q$ for each q. We then apply the global section functor $\Gamma(X, -)$ to $\mathscr{C}^{**}$ and get the double complex $D^{**} = \Gamma(X, \mathscr{C}^{**})$. The n-th Čech hypercohomology group $\check{\mathbb{H}}^n_G(\mathscr{U}, \mathcal{F}^*)$ is by definition the n-th cohomology group of the total complex of D^{**}. For details, see [9, Section 3.4] and [22, Chapter 5].

For a general G-space, we can not expect too much. However, when we apply the $\mathrm{RO}(G)$-graded cohomology theory to a smooth G-manifold X, things are different. One of the reasons is that the above theorem guarantees the existence on X of a cofinal system consisting of equivariant good covers. In Section 4 we show that with some proper coefficient system, there is an isomorphism between the $\mathrm{RO}(G)$-graded cohomology groups and the Čech hypercohomology groups associated to each equivariant good cover $\mathscr{U}$ of X.

It proceeds as follows: Let M be a discrete $\mathbb{Z}[G]$-module and let $\underline{M}$ be the coefficient system associated to M ([2]). For a finite dimensional representation V of G we define a cochain complex of presheaves $M(V)$ (Section 3). By constructing two spectral sequences converging to $\mathrm{RO}(G)$-graded cohomology groups and Čech hypercohomology groups, respectively, we show that, for any equivariant good cover $\mathscr{U}$ of X and $n \in \mathbb{Z}$, there is a natural isomorphism $\check{\mathbb{H}}^n_G(\mathscr{U}, M(V)) \cong H^{V+n-\dim(V)}_{\mathrm{Br}}(X, \underline{M})$. Since the equivariant good covers are cofinal among all open covers of X, we have the following main theorem.

Theorem. *(See Theorem 4.5.) Let X be a based smooth G-manifold. Then there is a natural isomorphism*

$$\check{\mathbb{H}}^n_G\big(X, M(V)\big) \cong \widetilde{H}^{V+n-\dim(V)}_{\mathrm{Br}}(X, \underline{M}),$$

where $\check{\mathbb{H}}^n_G(X, \underline{M})$ is the direct limit of the system $\check{\mathbb{H}}^n_G(\mathscr{U}, \underline{M})$ over equivariant good covers $\mathscr{U}$ on X.