Journal of Algebra 412 (2014) 51–86

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Jordan quadruple systems ☆

Murray Bremner *, Sara Madariaga

Department of Mathematics and Statistics, University of Saskatchewan, Canada

A R T I C L E   I N F O

Article history:
Received 6 March 2014
Available online 17 May 2014
Communicated by Jean-Yves Thibon

To our friend and colleague Luiz Antonio Peresi on his retirement

MSC:
primary 17C05
secondary 17A42, 17C50, 17C55, 18D50

Keywords:
Jordan tetrad
Polynomial identities
Gröbner bases
Universal associative envelopes
Wedderburn decompositions

A B S T R A C T

We define Jordan quadruple systems by the polynomial identities of degrees 4 and 7 satisfied by the Jordan tetrad $\{a, b, c, d\} = abcd + dcba$ as a quadrilinear operation on associative algebras. We find further identities in degree 10 which are not consequences of the defining identities. We introduce four infinite families of finite dimensional Jordan quadruple systems, and construct the universal associative envelope for a small system in each family. We obtain analogous results for the anti-tetrad $[a, b, c, d] = abcd - dcba$. Our methods rely on computer algebra, especially linear algebra on large matrices, the LLL algorithm for lattice basis reduction, representation theory of the symmetric group, noncommutative Gröbner bases, and Wedderburn decompositions of associative algebras.

1. Introduction

In this paper we study the quadrilinear operations $\{a, b, c, d\} = abcd + dcba$ and $[a, b, c, d] = abcd - dcba$ in associative algebras. The first is the Jordan tetrad which

☆ Murray Bremner was supported by a Discovery Grant from NSERC, the Natural Sciences and Engineering Research Council. Sara Madariaga was supported by a Postdoctoral Fellowship from PIMS, the Pacific Institute for the Mathematical Sciences.

* Corresponding author.
E-mail addresses: bremner@math.usask.ca (M. Bremner), madariaga@math.usask.ca (S. Madariaga).

http://dx.doi.org/10.1016/j.jalgebra.2014.05.001
0021-8693/

plays an important role in the structure theory of Jordan algebras [26,29]. The second is the anti-tetrad, which seems not to have been studied until now.

1.1. Motivation

In an associative algebra for $n \geq 2$ we define the n-tad to be this n-ary multilinear operation: $\{a_1, \ldots, a_n\} = a_1 \cdots a_n + a_n \cdots a_1$. For $n = 2$ we obtain the Jordan product $\{a, b\} = ab + ba$ satisfying commutativity and the Jordan identity:

$$\{a, b\} \equiv \{b, a\}, \qquad \{\{\{a, a\}, b\}, a\} \equiv \{\{a, a\}, \{b, a\}\}.$$

There are further "special" identities satisfied by the Jordan product in every associative algebra which do not follow from the defining identities; the simplest occur in degrees 8 and 9 and are called the Glennie identities [17,18]. A Jordan algebra is "special" if it can be represented as a subspace of an associative algebra closed under the Jordan product; otherwise, it is "exceptional". If a special Jordan algebra is finite dimensional then its universal associative enveloping algebra is also finite dimensional. A survey of the role of identities in Jordan theory has been given by McCrimmon [24]. For the structure and representation theory of finite dimensional Jordan algebras, see Jacobson [19]. For the modern theory including infinite dimensional algebras, see McCrimmon [25].

For $n = 3$ we obtain the Jordan triple product $abc + cba$; in every associative algebra, this operation satisfies identities which define Jordan triple systems (JTS):

$$\begin{gathered}\{a, b, c\} \equiv \{c, b, a\}, \\ \{\{a, b, c\}, d, e\} \equiv \{\{a, d, e\}, b, c\} - \{a, \{b, e, d\}, c\} + \{a, b, \{c, d, e\}\}.\end{gathered}$$

In contrast to the Jordan identity these identities are multilinear. There are special identities in higher degree: identities satisfied by the Jordan triple product in every associative algebra but which do not follow from the defining identities [22,23]. For the classification of finite dimensional JTS, see [20,27,28] and for their universal associative envelopes, see [21].

Closely related to Jordan triple systems are the anti-Jordan triple systems (AJTS), see [16]. Finite dimensional simple AJTS have been classified [1]. These systems are defined by identities satisfied by the anti-Jordan triple product $abc - cba$ in every associative algebra:

$$\begin{gathered}\{a, b, c\} + \{c, b, a\} \equiv 0, \\ \{\{a, b, c\}, d, e\} \equiv \{\{a, d, e\}, b, c\} + \{a, \{b, e, d\}, c\} + \{a, b, \{c, d, e\}\}.\end{gathered}$$

Universal associative envelopes for one infinite family of simple AJTS have been constructed [15].