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On faithfully flat fibrations by a punctured line

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ABSTRACT

Let R be a Noetherian normal domain. In this paper, we present the structure of a faithfully flat R-algebra A whose generic and codimension one fibres are of the form $K[X, \frac{1}{f(X)}]$. We also investigate minimal sufficient conditions for such an algebra to be finitely generated over R.

1. Introduction

Let R be an integral domain and A an R-algebra such that $\text{tr.deg}_R(A) = 1$. Recall that for a prime ideal P in R of height r, $k(P)$ denotes the residue field R_P/PR_P and $A \otimes_R k(P)$ is called *the fibre ring (or fibre) of codimension* r at the point P of $\operatorname{Spec} R$. We say that the fibre ring at $P \in \operatorname{Spec} R$ "misses n points" if $A \otimes_R k(P)$ is of the form $k(P)[X, \frac{1}{(X-\alpha_1)\cdots(X-\alpha_n)}]$ for some indeterminate X and n distinct elements

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$\alpha_1, \dots, \alpha_n \in k(P)$. Thus, if the fibre ring is $\mathbb{A}^*$ (i.e., isomorphic to the Laurent polynomial ring $k(P)[X, \frac{1}{X}]$), then we say that it misses one point.

It is known that if R is a Noetherian normal domain and A a faithfully flat R-algebra whose fibre rings are all $\mathbb{A}^*$, then $A \cong \bigoplus_{n\in\mathbb{Z}} I^n$ for some invertible ideal I of R ([1, Theorem 3.11], [3, Theorem 4.8]). Thus, if R is a UFD, then A is itself $\mathbb{A}^*$. R.V. Gurjar asked the author to explore the structure of $\mathbb{A}^{**}$-fibrations (i.e., algebras whose fibres miss two points). We consider a general version of his question.

Q.1. Let R be a Noetherian normal domain. What can we say about the structure and properties of a faithfully flat R-algebra A each of whose fibre rings misses at least one point and some of the fibre rings miss at least two points?

We also consider a closely related problem. In [2], an R-algebra B has been defined to be *locally* $\mathbb{A}^1$ *in codimension-one (over R)* if $B_P (= B \otimes_R R_P)$ is a polynomial algebra in one variable over R_P for every height one prime ideal P of R. The structure and properties of a faithfully flat algebra B over a Noetherian normal domain R which is locally $\mathbb{A}^1$ in codimension-one have been extensively studied by Bhatwadekar, Dutta and Onoda in [2]. Bhatwadekar and Gupta have described in [3] the structure and properties of a faithfully flat algebra A over a Noetherian normal domain R satisfying $A_P \cong R_P[X, \frac{1}{a_P X + b_P}]$ with $a_P, b_P \in R_P$, for every height one prime ideal P of R. They showed that the R-algebra A is of the form $B[I^{-1}]$, where B is faithfully flat and locally $\mathbb{A}^1$ in codimension-one over R and I is an invertible ideal of B. These results lead us to the more general question (first raised to the author by A.K. Dutta):

Q.2. Let R be a Noetherian normal domain. What can one say about the structure and properties of a faithfully flat R-algebra A satisfying the condition $A_P \cong R_P[X, \frac{1}{f_P(X)}]$ with $f_P(X) \in R_P[X]$ for every height one prime ideal P of R?

In this paper we investigate the above two questions. Q.2 is addressed in Theorem A below (Theorem 5.2):

Theorem A. *Let R be a Noetherian normal domain with field of fractions K and A be a faithfully flat R-algebra such that:*

(i) *$A \otimes_R K = K[X, \frac{1}{f(X)}]$ for some $f(X) \in K[X] \setminus K$ and X transcendental over K.*
(ii) *For each prime ideal P in R of height one, $A_P \cong R_P[X, \frac{1}{f_P(X)}]$ for some $f_P(X) \in R_P[X]$.*

Let $B := A \cap K[X]$. Then B is a faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one over R and there exists an invertible ideal I of B such that $A = B[I^{-1}]$. Moreover, A is Noetherian (respectively, finitely generated over R) if and only if B is Noetherian (respectively, finitely generated over R).