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On finite solvable groups whose cyclic p-subgroups of equal order are conjugate

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ABSTRACT

We study the finite solvable groups G whose cyclic p-subgroups of the same order are conjugate in G whenever p is a prime number dividing the order of G.

1. Introduction

It is our purpose to clarify the structure of the solvable finite groups whose cyclic subgroups of equal prime-power order are conjugate. In [3], M. Costantini and E. Jabara did show what structure a finite group necessarily must have, in order that G might satisfy the property that any two cyclic subgroups of the same order are conjugate. Clearly, every group sharing the property in [3] satisfies our property (P) but the converse implication of these properties does not hold. Notation is standard or self-explanatory; see [1], [4], and [6]. All groups in this paper will be finite. The 1-dimensional group $\mathrm{Aff}(1,q)$ consists of semilinear maps $\xi \mapsto \alpha\xi^{\sigma}$ where $\alpha \in \mathrm{GF}(q)^{\times}$ and $\sigma \in \mathrm{Gal}(\mathrm{GF}(q))$. Note that $\mathrm{Aff}(1,q)$ is metacyclic.

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Definition. A finite group G is a *(P)-group* if every two cyclic subgroups of prime-power order are conjugate in G. In short we say that such G satisfy property (P). A (P)-group G is *(P)-subdirectly irreducible* if it does not have two minimal normal subgroups of relatively prime orders.

There are two well-known families of groups which are (P)-groups, namely metacyclic groups and $\mathrm{PSL}(2,q)$. All cyclic groups are (P)-groups, as well as all abelian (P)-groups are cyclic. Moreover the nilpotent groups, the dihedral groups of order $2m$ with m odd, $\mathrm{Alt}(4)$ but not $\mathrm{Sym}(4)$, $\mathrm{Aff}(1,p^d)$, $\mathrm{SL}(2,3)$, $\mathrm{PSL}(2,7)$ are examples of (P)-groups. These examples will play an important role later. Since (P) is a very strong condition, we doubt that a random group is embeddable in a (P) group. We show that except in one of three cases from Theorem (6.1) due to B. Huppert, most solvable (P)-groups are very large subgroups of $V \rtimes \mathrm{Aff}(1,p^d)$, where $V = \mathrm{GF}(p)^d$ is a vector space of dimension d over the field $\mathrm{GF}(p)$.

Due to Theorem (2.8) it is of fundamental importance to elucidate the structure of (P)-subdirectly irreducible groups. As to solvable such groups, we will get as main Theorem A as well as some (important) additional Theorems B and C.

Theorem A. *Let G be solvable group. Then G is a (P)-subdirectly irreducible group if and only if one of the following holds*

(1) *$G = A \rtimes B$ where $A = \mathrm{GF}(p)^d$ is elementary abelian of order p^d, $p \nmid |B|$, B being a (P)-group whose Sylow subgroups are all cyclic, acting transitively on the set of subgroups of A of order p;*

(2) *$G = A \rtimes B$ where either $A = \mathrm{GF}(5)^2$ and $B \cong \mathrm{SL}(2,3)$ or $A = \mathrm{GF}(11)^2$ and $B \cong \mathrm{SL}(2,3) \times C_a$ with $a = 1$ or $a = 5$. In this case, B acts transitively on the set of subgroups of A of order 5 and we see that $G''' \le A \le \mathbb{F}(G)$ and $G^{(4)} = 1$;*

(3) *either $G = \mathrm{SL}(2,3)$ or G is a (P)-group whose Sylow 2-subgroups are Suzuki 2-groups.*

As to related but sometimes very different kind of research the reader should consult the papers [3,5,9,10,12,14].

2. Preliminaries

(2.1) Lemma. *Let N be a normal subgroup of a (P)-group G. Let p be a prime and suppose that the maximal possible order of a p-element in N is p^a. If x is a p-element of G of order p^b and not contained in N, then $b > a$ and $\bar{x} = xN$ has order p^{b-a} in G/N.*

Proof. By assumption, N contains elements of order $p, p^2, \cdots, p^a$. If y is an element of G having order p^c with $c \le a$, then y has the same order as an element of N. Thus, by property (P), $\langle y \rangle$ is conjugate to the subgroup generated by this element of N and