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Algorithms and topology of Cayley graphs for groups

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ABSTRACT

Autostackability for finitely generated groups is defined via a topological property of the associated Cayley graph which can be encoded in a finite state automaton. Autostackable groups have solvable word problem and an effective inductive procedure for constructing van Kampen diagrams with respect to a canonical finite presentation. A comparison with automatic groups is given. Another characterization of autostackability is given in terms of prefix-rewriting systems. Every group which admits a finite complete rewriting system or an asynchronously automatic structure with respect to a prefix-closed set of normal forms is also autostackable. As a consequence, the fundamental group of every closed 3-manifold with any of the eight possible uniform geometries is autostackable.

1. Introduction

A primary motivation for the definition of the class of automatic groups is to make computing the word problem for 3-manifold groups tractable; however, in their

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introduction of the theory of automatic groups, Epstein et al. [10] showed that the fundamental group of a closed 3-manifold having Nil or Sol geometry is not automatic. Brady [1] showed that there are Sol geometry groups that do not belong to the wider class of asynchronously automatic groups. Bridson and Gilman [4] further relaxed the language theoretic restriction on the associated normal forms, replacing regular with indexed languages, and showed that every 3-manifold group has an asynchronous combing with respect to an indexed language. More recently, Kharlampovich, Khoussainov, and Miasnikov [22] have defined the class of Cayley automatic groups, extending the notion of an automatic structure (preserving the regular language restriction), but it is as yet unknown whether all Nil and Sol 3-manifold groups are Cayley automatic. In this paper we define the notion of autostackability for finitely generated groups using properties very closely related to automatic structures, that holds for 3-manifold groups of all uniform geometries.

Let G be a group with an inverse-closed finite generating set A, and let $\Gamma = \Gamma(G, A)$ be the associated Cayley graph. Let $\vec{E}$ be the set of directed edges; for each $g \in G$ and $a \in A$, let $e_{g,a}$ denote the directed edge of Γ with initial vertex g, terminal vertex ga, and label a. Let $\mathcal{N} \subset A^*$ be a set of normal forms for G over A; for each $g \in G$, we denote the normal form word representing g by y_g. Note that whenever we have an equality of words $y_g a = y_{ga}$ or $y_g = y_{ga} a^{-1}$, then there is a van Kampen diagram for the word $y_g a y_{ga}^{-1}$ that contains no 2-cells; in this case we call the edge $e_{g,a}$ *degenerate*. Let $\vec{E}_{\mathcal{N},d} = \vec{E}_d$ be the set of all degenerate directed edges, and let $\vec{E}_{\mathcal{N},r} = \vec{E}_r := \vec{E} \setminus \vec{E}_d$; we refer to elements of $\vec{E}_r$ as *recursive* edges.

Definition 1.1. A group G with finite inverse-closed generating set A is *autostackable* if there are a set $\mathcal{N}$ of normal forms for G over A containing the empty word, a constant k, and a function $\phi : \mathcal{N} \times A \to A^*$ such that the following hold:

(1) The graph of the function ϕ,

$$graph(\phi) := \{(y_g, a, \phi(y_g, a)) \mid g \in G, a \in A\},$$

is a synchronously regular language.

(2) For each $g \in G$ and $a \in A$, the word $\phi(y_g, a)$ has length at most k and represents the element a of G, and:
  - (2d) If $e_{g,a} \in \vec{E}_{\mathcal{N},d}$, then the equality of words $\phi(y_g, a) = a$ holds.
  - (2r) The transitive closure $<_\phi$ of the relation $<$ on $\vec{E}_{\mathcal{N},r}$, defined by

    $e' < e_{g,a}$ whenever $e_{g,a}, e' \in \vec{E}_{\mathcal{N},r}$ and e' is on the directed path in Γ labeled $\phi(y_g, a)$ starting at the vertex g,

    is a strict well-founded partial ordering.

Removing the algorithmic property in (1), the group G is called *stackable* over the inverse-closed generating set A if property (2) holds for some normal form set $\mathcal{N}$