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Computing Hasse–Schmidt derivations and Weil restrictions over jets

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ABSTRACT

We give an explicit and compact description of the Hasse–Schmidt derivations, using Fitting ideals and symmetric tensor algebras. Finally we verify the localization conjecture [9].

1. Introduction

The higher order derivations were introduced by Hasse and Schmidt, and generalize the usual notion of derivations. We will give a new description of the higher order derivations, a description using Fitting ideals and symmetric tensor algebras. We obtain a compact presentation of the higher order derivations, naturally relating the order n derivations with those of order $n-1$. Finally we verify that the Hasse–Schmidt derivations behave well under localization, thereby showing that the localization conjecture [9] holds.

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Let R and A' be two (commutative) A-algebras. A higher order derivation [5], of order n, is a sequence $\partial_n = (d_0, \dots, d_n)$ of A-linear maps $d_p: R \longrightarrow A'$, where d_0 is an algebra homomorphism, and where

$$d_p(xy) = \sum_{i+j=p} d_i(x)d_j(y),$$

for all $1 \leq p \leq n$. A higher order derivation is also called Hasse–Schmidt derivation, and works relating these to ordinary derivations can be found in e.g. [7,3].

We let $\mathrm{Der}^n_{R/A}(A')$ denote the set of all higher order derivations from R to A'. By composition one obtains that the set of higher order derivations form a functor $\mathrm{Der}^n_{R/A}(-)$, from the category of A-algebras to sets. One can construct the representing object $\mathrm{HS}^n_{R/A}$ by forming the polynomial ring over R, with n variables for each element in R, modulo all the expected relations, see e.g. [10].

We will give a different, and a more compact, description of the representing object $\mathrm{HS}^n_{R/A}$. It is well-known that Hasse–Schmidt derivations are equivalently described by jets and arc spaces. An A-algebra homomorphism

$$u: R \longrightarrow A'[t]/\left(t^{n+1}\right) = A' \bigotimes_A A[t]/\left(t^{n+1}\right)$$

encodes the same information as a higher order derivation from R to A'. In other words the higher order derivations are given as the functor $\mathrm{Hom}_{A\text{-alg}}(R, E)$, with co-domain the arc of jets $E = A[t]/(t^{n+1})$.

The functors $\mathrm{Hom}_{A\text{-alg}}(R, E)$ were described explicitly using Fitting ideals in [8], and the results in the present paper are obtained by specializing to $E = A[t]/(t^{n+1})$. With this specific co-domain $E = A[t]/(t^{n+1})$ we will write down explicitly the representing object, and use the explicit presentation to extract new information as well as answer questions about the Hasse–Schmidt derivations.

Our explicit presentation is as follows. Write the A-algebra $R = A[x, y, \dots, z]/(f_1, \dots, f_m)$, that is as a quotient of a polynomial ring modulo some relations (for notational simplicity we present the result only using finitely many variables and relations). Then the Hasse–Schmidt derivations of order n are represented by $\mathrm{HS}^n_{R/A}$ which is the algebra

$$\mathrm{HS}^{n-1}_{R/A}[d_n x, d_n y, \dots, d_n z]/(d_n f_1, \dots, d_n f_m),$$

where $d_n x, d_n y, \dots, d_n z$ are variables over $\mathrm{HS}^{n-1}_{R/A}$, and where $d_n f$ is the n'th order derivation of $f \in A[x, y, \dots, z]$. In particular we recover that $\mathrm{HS}^1_{R/A}$ equals the symmetric tensor algebra of the R-module of differentials $\Omega^1_{R/A}$.

As the Weil restriction of $A[t]/(t^{n+1})$-algebras is closely related, we also write explicit presentations for these rings. The Weil restrictions are important objects within algebraic geometry, and it might be of interest to see explicit presentations of these algebras as well.