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p-Parts of character degrees and the index of the Fitting subgroup ☆

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ABSTRACT

In a solvable group G, if p^2 does not divide $\chi(1)$ for all $\chi \in \mathrm{Irr}(G)$, then we prove that $|G : F(G)|_p \leq p^2$. This bound is best possible.

1. Introduction

Let G be a finite group, let $\mathrm{Irr}(G)$ be the set of the irreducible complex characters of G and let p be a prime. Let P be a Sylow p-subgroup of G. The fundamental Itô–Michler

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Theorem asserts that p does not divide $\chi(1)$ for all $\chi \in \mathrm{Irr}(G)$ if and only if $P \lhd G$ and P is abelian. In particular, when this happens, we have that $G/F(G)$ has order not divisible by p, where $F(G)$ is the Fitting subgroup of G.

The Itô–Michler Theorem has been generalized in several ways (see [6] for a survey), and in this paper we propose one further direction. Suppose that p^2 does not divide $\chi(1)$ for all $\chi \in \mathrm{Irr}(G)$. Is it possible to bound the largest power of p dividing $|G : F|$?

Theorem 1. *Let G be a finite solvable group and suppose that p^2 does not divide $\chi(1)$ for all $\chi \in \mathrm{Irr}(G)$. Then $|G : F(G)|_p \leq p^2$. In particular, if $P \in \mathrm{Syl}_p(G)$, then P' is subnormal in G.*

Previous works [1,5,7] give bounds of $|G : F(G)|_p$ by p^8, for solvable G, and even p^2 (but only when $|G|$ is odd). It is an interesting problem to decide if there is a version of Theorem 1 for non-solvable groups. Using Theorem 1 and the main result of [3], at least we can settle the $p = 2$ case.

Corollary 2. *Let G be a finite group. If 4 does not divide $\chi(1)$ for all $\chi \in \mathrm{Irr}(G)$, then $|G : F(G)|_2 \leq 8$. In particular, if $P \in \mathrm{Syl}_2(G)$, then P'' is subnormal in G.*

The alternating group A_7 shows that the bound in the previous corollary is best possible.

We conclude this introduction by noting that Yong Yang has considered a similar question. In [9], he has proved that if p^a is the largest power of p dividing $\chi(1)$ for $\chi \in \mathrm{Irr}(G)$, then $|G : F(G)|_p \leq p^{3a}$ when $p \geq 5$ is a prime. In fact, the method of Theorem B of [9] can be used to improve this bound to $p^{2.5a}$ when $p \geq 5$. This yields our result when $p \geq 5$. However, the main difficulty in proving our result occurs when $p = 2$.

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2. Preliminaries

If G is a group, we denote by $1 = F_0(G) \leq F_1(G) = F(G) \leq F_2(G) \leq \ldots$ the Fitting series of G. That is, $F_{i+1}(G)/F_i(G) = F(G/F_i(G))$ for integers $i > 0$. We start with the following.

Lemma 2.1. *Suppose that p^2 does not divide $\chi(1)$ whenever $\chi \in \mathrm{Irr}(G)$ for a solvable group G. Then the following are true:*