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## Preordered forests, packed words and contraction algebras



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### ABSTRACT

We introduce the notions of preordered and heap-preordered forests, generalizing the construction of ordered and heap-ordered forests. We prove that the algebras of preordered and heap-preordered forests are Hopf for the cut coproduct, and we construct a Hopf morphism to the Hopf algebra of packed words. Moreover, we define another coproduct on the preordered forests given by the contraction of edges. Finally, we give a combinatorial description of morphism defined on Hopf algebras of forests with values in the Hopf algebras of shuffles or quasi-shuffles.

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### R É S U M É

Nous introduisons les notions de forêts préordonnées et préordonnées en tas, généralisant les constructions des forêts ordonnées et ordonnées en tas. On démontre que les algèbres des forêts préordonnées et préordonnées en tas sont des algèbres de Hopf pour le coproduit de coupes et on construit un morphisme d'algèbres de Hopf dans l'algèbre des mots tassés. D'autre part, nous définissons un autre coproduit sur les forêts préordonnées donné par la contraction d'arêtes. Enfin, nous donnons une description combinatoire de morphismes définis sur des algèbres de Hopf de forêts et à valeurs dans les algèbres de Hopf de battages et de battages contractants.

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**Introduction**

The Connes–Kreimer Hopf algebra of rooted forests  $\mathbf{H}_{CK}$  is introduced and studied in [2,18]. This commutative, noncocommutative Hopf algebra is used to study a problem of Renormalization in Quantum Field Theory, as explained in [3,4]. The coproduct is given by admissible cuts. We denote by  $\mathbf{H}_{CK}^{\mathcal{D}}$  the Hopf algebra of rooted trees, where the vertices are decorated by decorations belonging to the set  $\mathcal{D}$ . A noncommutative version, the Hopf algebra  $\mathbf{H}_{NCK}$  of planar rooted forests, is introduced in [7,8,14]. When the vertices are given a total order, we obtain the Hopf algebra of ordered forests  $\mathbf{H}_o$  and, adding an increasing condition, we obtain the Hopf subalgebra of heap-ordered forests  $\mathbf{H}_{ho}$  (see [11,12]).

On the other side, M.E. Hoffman studied in [13] the Hopf algebra of shuffles  $\mathbf{Sh}^{\mathcal{D}}$  and the Hopf algebra of quasi-shuffles  $\mathbf{Csh}^{\mathcal{D}}$ . In Mould Calculus theory, J. Ecalle and B. Vallet constructed a Hopf algebra morphism from  $\mathbf{H}_{CK}^{\mathcal{D}}$  to  $\mathbf{Sh}^{\mathcal{D}}$ , or  $\mathbf{Csh}^{\mathcal{D}}$ , called the arborification morphism, or the contracting arborification morphism (see [6]). These morphisms are used for example in the problem of normal forms for vector fields to prove the convergence of series. We will describe in this paper all morphisms from  $\mathbf{H}_{CK}^{\mathcal{D}}$  to  $\mathbf{Sh}^{\mathcal{D}}$  or  $\mathbf{Csh}^{\mathcal{D}}$ .

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