Journal of Algebra 411 (2014) 337–355

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Fibers of flat morphisms and Weierstrass preparation theorem ☆

Đoàn Trung Cường [a,b,*]

[a] *Vietnam Institute for Advanced Study in Mathematics, Ta Quang Buu Building, 01 Dai Co Viet, Hai Ba Trung, Hanoi, Viet Nam* [1]
[b] *Institute of Mathematics, 18 Hoang Quoc Viet, 10307 Hanoi, Viet Nam* [2]

A R T I C L E   I N F O

Article history:
Received 5 February 2014
Available online 21 May 2014
Communicated by Kazuhiko Kurano

Dedicated to Professor Nguyễn Tự Cường

MSC:
11E08
11E12
13B35
14H05

Keywords:
Fiber of flat morphism
Weierstrass preparation theorem
Weierstrass extension
u-Invariant

A B S T R A C T

We characterize flat extensions of commutative rings satisfying the Weierstrass preparation theorem. Using this characterization we prove a variant of the Weierstrass preparation theorem for rings of functions on a normal curve over a complete local domain of dimension one. This generalizes recent works of Harbater, Hartmann and Krashen with a different method of proof.

☆ This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.01-2012.05.
* Correspondence to: Institute of Mathematics, 18 Hoang Quoc Viet, 10307 Hanoi, Viet Nam.
E-mail address: doantc@gmail.com.
[1] Current address.
[2] Permanent address.

http://dx.doi.org/10.1016/j.jalgebra.2014.04.014
0021-8693/

0. Introduction

The Weierstrass preparation theorem is an important result in the theory of several complex variables. Its main idea is that the local behavior of a holomorphic function is similar to the behavior of a polynomial [4]. There are several ways to generalize this theorem for different purposes in other contexts. The algebraic form of the Weierstrass preparation theorem has important applications in algebra and algebraic number theory. For example, the usage of the algebraic form in the study of quadratic forms is well-known (cf. [2]).

Recently Harbater–Hartmann–Krashen have generalized the algebraic form of the Weierstrass preparation theorem to functions on curves over a complete discrete valuation ring. Using essentially techniques of patching, they showed that a fraction of elements in certain completions of the ring of regular functions on the curve factors as product of a rational function and an invertible power series [7–9]. This result was then applied to study quadratic forms and central simple algebras over the related fields of functions. A significant application is to show that a field extension of transcendence degree one over the p-adic numbers has u-invariant 8. This is one of three (recent) proofs for this long-standing problem (see [3,8,11,16]).

The aim of the current paper is to generalize further the theorem of Harbater–Hartmann–Krashen on the Weierstrass preparation theorem with a different approach through ring theory and algebraic geometry. Weierstrass preparation is a kind of factorization assertion for elements of a ring extension. If the extension under consideration satisfies the Weierstrass preparation theorem (in the sense of Definition 1.1 in Section 1) then all the fibers have dimension zero. This observation is our starting point. In the works of Harbater, Hartmann and Krashen, the authors consider a normal curve over a complete discrete valuation ring and study the functions that are rational on a subset of an irreducible component of the closed fiber. In our approach, we consider functions that are regular on an arbitrary subset of the closed fiber without restriction to a component. A careful analysis on the fibers of flat extensions allows us to treat the branching phenomenon when passing from functions on a component to the general case. It also gives another proof for the generalized Weierstrass preparation theorem of Harbater–Hartmann–Krashen. It is remarkable that we mainly work on the level of rings rather than on the fields.

About the structure of the paper, we introduce in Section 1 a notion of Weierstrass homomorphism and recall an application of the Weierstrass preparation theorem in the study of the u-invariant of a field. In Section 2 we consider flat homomorphisms which are Weierstrass. The main result of the section is Theorem 2.3 presenting a characterization of such a Weierstrass flat homomorphism via properties of the fibers and the residue fields. This characterization is used in Section 3 to generalize the Weierstrass preparation theorem for functions on curves over a complete local domain of dimension one (see Theorems 3.3, 3.5, 3.6). A consequence of the characterization theorem in Section 2 is that over a local ring, a henselian Weierstrass extension is (up to isomorphism)