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Homotopy bases and finite derivation type for subgroups of monoids

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ABSTRACT

Given a monoid defined by a presentation, and a homotopy base for the derivation graph associated to the presentation, and given an arbitrary subgroup of the monoid, we give a homotopy base (and presentation) for the subgroup. If the monoid has finite derivation type (FDT), and if under the action of the monoid on its subsets by right multiplication the strong orbit of the subgroup is finite, then we obtain a finite homotopy base for the subgroup, and hence the subgroup has FDT. As an application we prove that a regular monoid with finitely many left and right ideals has FDT if and only if all of its maximal subgroups have FDT. We use this to show that a finitely presented regular monoid with finitely many left and right ideals satisfies the homological finiteness condition FP_3 if all of its maximal subgroups satisfy the condition FP_3.

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1. Introduction

Geometric methods play an important role in combinatorial group theory (see [16]). More recently, in [28] Squier (and in independent work Pride [23], and Kilibarda [13]) has developed a homotopy theory for monoids. Given a monoid presentation, Squier constructs a graph, called a *derivation graph*, and then defines certain equivalence relations called *homotopy relations* on the set of paths in the graph. A set of closed paths that generates the full homotopy relation is called a *homotopy base*. A monoid defined by a finite presentation is said to have *finite derivation type* (FDT) if there is a finite homotopy base for the presentation. Squier showed that FDT is a property of the monoid, in the sense that it is independent of the choice of (finite) presentation.

The homotopical finiteness condition FDT was originally introduced as a tool for the study of finite string rewriting systems. It was shown in [28] that if a monoid is defined by a finite complete rewriting system then that monoid has FDT. In [14] Kobayashi showed that every one-relator monoid has FDT; it is still an open question whether every one-relator monoid admits a finite complete rewriting system. More background on the relationship with finite complete rewriting systems may be found in [22].

There are also important connections to the homology theory of monoids. If a monoid has FDT then it must satisfy the homological finiteness condition FP_3, and for groups FDT and FP_3 are equivalent; see [4,23,15,5]. In [24] it was shown that for monoids FDT and the homological finiteness condition FHT (in the sense of [32]) are not equivalent. In addition to this, Squier's homotopy theory has been applied in [10] to the study of *diagram groups*, which are precisely fundamental groups of Squier complexes of monoid presentations.

One important area in the study of finiteness conditions is the consideration of their closure properties. Given a finiteness condition how does the property holding in a monoid relate to it holding in the substructures of that monoid (and vice versa)? For groups when passing to finite index subgroups, or finite index extensions, the properties of being finitely generated, finitely presented, having FP_n, or having FDT are all preserved; see [17,2]. Results about the behavior of FDT for monoids, when passing to submonoids and extensions appear in [30,18,24,21,31,20].

In this paper we investigate the relationship between the property FDT holding in a monoid, and the property holding in the subgroups of that monoid. Results of this type have been obtained for the properties of being finitely generated and presented (see [26]), and for being residually finite (see [6]). Specifically, given a monoid S defined by a presentation $\mathcal{P}_S$, given an associated homotopy base X, and given a subgroup G of S, we show how to obtain a homotopy base Y for a presentation $\mathcal{P}_G$ of G (this will be done in Section 3). Moreover, when the presentation $\mathcal{P}_S$ and the homotopy base X are both finite, and G has only finitely many *cosets* in S (where a coset of G is a set Gs, with $s \in S$, such that there exists $s' \in S$ with $Gss' = G$) then the presentation $\mathcal{P}_G$ and homotopy base Y will both be finite. A subgroup G of a monoid S with finitely many cosets, in the above sense, is said to have finite *translational index*. This is equivalent to