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The p-adic group ring of $\mathrm{SL}_2(p^f)$

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ABSTRACT

In the present article we show that the $\mathbb{Z}_p[\zeta_{p^f-1}]$-order $\mathbb{Z}_p[\zeta_{p^f-1}]\,\mathrm{SL}_2(p^f)$ can be recognized among those orders whose reduction modulo p is isomorphic to $\mathbb{F}_{p^f}\,\mathrm{SL}_2(p^f)$ using only ring-theoretic properties. In other words we show that $\mathbb{F}_{p^f}\,\mathrm{SL}_2(p^f)$ lifts uniquely to a $\mathbb{Z}_p[\zeta_{p^f-1}]$-order, provided certain reasonable conditions are imposed on the lift. This proves a conjecture made by Nebe in [8] concerning the basic order of $\mathbb{Z}_2[\zeta_{2^f-1}]\,\mathrm{SL}_2(2^f)$.

1. Introduction

Let p be a prime and let $(K, \mathcal{O}, k)$ be a p-modular system. This article is concerned with the group ring $\mathcal{O}\,\mathrm{SL}_2(p^f)$, where $f \in \mathbb{N}$. Hence we are dealing with the discrete valuation ring version of what is typically referred to as representation theory in "defining characteristic". Our aim in this paper is to prove a conjecture made by Nebe in [8] which claims to describe the group ring of $\mathrm{SL}_2(2^f)$ over sufficiently large extensions $\mathcal{O}$ of $\mathbb{Z}_2$. We are also interested in the question of whether the results in [9], which deal with the case $p \neq 2$, are sufficient to describe the group ring $\mathcal{O}\,\mathrm{SL}_2(p^f)$. Here, "to describe the group ring" means to describe its basic order. Our proof of Nebe's conjecture is indirect, and consists essentially of showing that a "unique lifting theorem" (see Corollary 7.15)

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holds for the group ring of $\mathrm{SL}_2(p^f)$. Basically this unique lifting theorem asserts that any $\mathcal{O}$-order reducing to $k\,\mathrm{SL}_2(p^f)$ which has semisimple K-span and is self-dual has to be isomorphic to $\mathcal{O}\,\mathrm{SL}_2(p^f)$. Note however that some details have been omitted in this short explanation. Namely, there are some technical conditions on the bilinear form with respect to which the $\mathcal{O}$-order is self-dual, and we also need to assume $k \supseteq \mathbb{F}_{p^f}$. Nebe's conjecture is an immediate consequence of this theorem, but the theorem may well be considered an interesting result in its own right.

This work is a continuation of the author's work in [4], where a "unique lifting theorem" similar to the one mentioned above is proved for 2-blocks with dihedral defect group. Our approach is, as in [4], based on the idea that, provided it is properly formulated, such a theorem holds for a k-algebra if and only if it holds for all k-algebras derived equivalent to the original one. By the abelian defect group conjecture, which is known to be true in the special case encountered in the present paper, the blocks of $k\,\mathrm{SL}_2(p^f)$ are derived equivalent to their Brauer correspondents. Technically, we must assume k to be algebraically closed for this, but we manage to work around that in this article. And, as it turns out, proving a "unique lifting theorem" for these Brauer correspondents is fairly easy due to their simple structure. In particular we prove Nebe's conjecture without ever having to put up with the complicated combinatorics that arises in the representation theory of $\mathrm{SL}_2(p^f)$.

The article is structured as follows: In Section 2 we introduce our notation and remind the reader of some basic definitions and facts on orders over discrete valuation rings. Section 3 gives a short summary of the results of [6,8] and [9]. In particular that section addresses the question of how our results extend the results of Nebe in [8] and [9], and actually lead to a complete description of the group ring $\mathcal{O}\,\mathrm{SL}_2(p^f)$. In Section 4 we explain how $\mathcal{O}$-orders reducing to a k-algebra A correspond to $\mathcal{O}$-orders reducing to a k-algebra B which is derived equivalent to A. This correspondence was introduced in [4], and we use it as a technical tool to deduce results about $\mathcal{O}$-orders reducing to the blocks of $k\,\mathrm{SL}_2(p^f)$ from analogous results about the Brauer correspondents of these blocks. Section 5 deals with the Brauer correspondents of the blocks of $k\,\mathrm{SL}_2(p^f)$. The main result of that section is Theorem 5.16, which is a unique lifting theorem for the aforementioned Brauer correspondents. Section 6 applies the correspondence of lifts introduced in Section 4 to the derived equivalence between the blocks of $k\,\mathrm{SL}_2(p^f)$ and their Brauer correspondents. This yields Corollary 6.4, which implies that a unique lifting theorem holds for $k\,\mathrm{SL}_2(p^f)$, where k is assumed to be algebraically closed. Section 7 deals with the case of non-algebraically closed fields k. Corollary 7.15 states a unique lifting theorem for the blocks of $k\,\mathrm{SL}_2(p^f)$ where $k \supseteq \mathbb{F}_{p^f}$. As an additional result we also obtain Corollary 7.17, which shows that there is a derived equivalence between the blocks of $\mathcal{O}\,\mathrm{SL}_2(p^f)$ and their Brauer correspondents for $\mathcal{O} = \mathbb{Z}_p[\zeta_{p^f-1}]$.