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On (almost) extreme components in Kronecker products of characters of the symmetric groups ☆

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Using a recursion formula due to Dvir, we obtain information on maximal and almost maximal components in Kronecker products of characters of the symmetric groups. This is applied to confirm a conjecture made by Bessenrodt and Kleshchev in 1999, which classifies all such Kronecker products with only three or four components.

1. Introduction

The decomposition of the tensor product of two representations of a group is an ubiquitous and notoriously difficult problem which has been investigated for a long time. For complex representations of a finite group this is equivalent to decomposing the Kronecker product of their characters into irreducible characters. An equivalent way

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of phrasing this problem for the symmetric groups is to expand the inner product of the corresponding Schur functions in the basis of Schur functions. Examples for such computations were done already a long time ago by Murnaghan and Littlewood (cf. [10,8]). While the answer in specific cases may be achieved by computing the scalar product of the characters, for the important family of the symmetric groups no reasonable general combinatorial formula is known.

Over many decades, a number of partial results have been obtained by a number of authors. To name just a few important cases that come up in the present article, the products of characters labeled by hook partitions or by two-row partitions have been computed (see [5,11,14]), and special constituents, in particular of tensor squares, have been considered [16,17]. For general products, the largest part and the maximal number of parts in a partition labeling a constituent of the product have been determined (see [3,4]); in fact, this is a special case of Dvir's recursion result [4, 2.3] that will be crucial in this paper. The recursion will be used to obtain information on components to partitions of maximal or almost maximal width and length, respectively.

In general, Kronecker products of irreducible representations have many irreducible constituents (see e.g. [7, 2.9]). In work by Kleshchev and the first author [1], situations are considered where the Kronecker product of two irreducible S_n-characters has few different constituents. It was shown there that such products are inhomogeneous (i.e., they contain at least two different irreducible constituents) except for the trivial situation where one of the characters is of degree 1; indeed, except for this trivial case, no constituent in a Kronecker product can simultaneously have maximal width and length. Investigating the question on homogeneous products for the representations of the alternating group A_n motivated the study of products of S_n-characters with two different constituents; all Kronecker products of irreducible S_n-characters with two homogeneous components were then classified in [1]. Also, some partial results for products with up to four homogeneous components were obtained, and we will use these here. Moreover, a complete classification of the pairs (χ, ψ) of irreducible complex S_n-characters such that the Kronecker product $\chi \cdot \psi$ has three or four homogeneous components was conjectured in [1]. In this article we obtain more precise information on the extreme and almost extreme constituents (and components, respectively) in Kronecker products, where the *extreme constituents* are labeled by partitions of maximal width or length, while the labels of the *almost extreme constituents* have width or length differing by one from the maximal width or length, respectively; by *(almost) extreme constituents (or components)* we mean constituents (or components) of both types. As a consequence of these results we confirm the conjectures just mentioned.

Note that $[\mu]$ denotes the irreducible complex character of S_n labeled by the partition μ of n; further details on notation and background can be found in Section 2.

Theorem 1.1. *Let μ, ν be partitions of n. Then the Kronecker product of the characters $[\mu]$ and $[\nu]$ has at least five (almost) extreme components unless we are in one of the following situations.*