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Invertibility preservers on central simple algebras

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ABSTRACT

We solve Kaplansky's problem concerning the structure of linear preservers of invertibility in the special case of maps on central simple algebras.

1. Introduction and statement of the main result

Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over a field $\mathbb{F}$. A linear map $\phi : \mathcal{A} \to \mathcal{B}$ preserves invertibility if $\phi(a) \in \mathcal{B}$ is invertible for every invertible element $a \in \mathcal{A}$. When treating such maps there is no loss of generality in assuming that they are unital. Indeed, for every such ϕ the map $a \mapsto \phi(1)^{-1}\phi(a)$, $a \in \mathcal{A}$, preserves invertibility and is unital.

The famous Gleason–Kahane–Żelazko theorem [6,8] states that every unital linear invertibility preserving map between commutative unital semisimple Banach algebras is multiplicative. Motivated by this result, Kaplansky posed the problem of studying such

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maps in his influential lecture notes [9]. As mentioned by Kaplansky, Banach algebras seem to be the natural setting for this problem. In this context the condition that a unital linear map ϕ preserves invertibility can be equivalently formulated as the assumption that it is spectrum compressing. It is then natural to consider also spectrum preserving maps, i.e., linear maps satisfying the stronger condition $\sigma(\phi(a)) = \sigma(a)$ for every $a \in \mathcal{A}$. It is easy to see that this property corresponds to the condition of preserving invertibility in both directions. A linear map $\phi : \mathcal{A} \to \mathcal{B}$ preserves invertibility in both directions if for every $a \in \mathcal{A}$ the element $\phi(a) \in \mathcal{B}$ is invertible if and only if a is invertible in $\mathcal{A}$. There is a vast literature on spectrum preserving and spectrum compressing maps as well as on related problems concerning maps preserving or compressing various parts of the spectrum, spectral radius isometries and spectrally bounded maps, etc. For more information we refer to the survey paper [1] where the following conjecture is formulated: every bijective linear unital map preserving invertibility (in both directions) between unital semisimple Banach algebras is a Jordan isomorphism. While the general solution of this problem seems to be out of reach at the moment, and is open even for general C^*-algebras, the conjecture has been confirmed for von Neumann algebras [2] and for the algebra of all bounded linear operators on a Banach spaces [14].

This kind of problems have been studied mainly from the functional analysis point of view. But, as already observed by Kaplansky [9], the first result in this direction was proved in purely algebraic setting. Already in 1959, Marcus and Purves [10] proved that every linear unital invertibility preserving map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is either an inner automorphism, or an inner anti-automorphism. Here, $M_n(\mathbb{C})$ denotes the algebra of all $n \times n$ complex matrices. At first glance it is a little bit surprising that the same result does not hold if we replace complex matrices by real matrices. Indeed, the map $\phi : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ defined by

$$\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

is unital, linear, and preserves invertibility (in one direction only), but is not an automorphism or an anti-automorphism. This example comes from the representation of complex numbers as 2×2 real matrices, and it is then easy to guess that one can construct similar examples in dimensions $n = 4$ and $n = 8$ using the matrix representations of quaternions and octonions, respectively.

Let $\mathbb{F}$ be an arbitrary field. It has been known that every bijective unital linear invertibility preserving map $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is either an inner automorphism, or an inner anti-automorphism (see [12] and the references therein). The same conclusion holds without the bijectivity assumption if we impose the stronger condition that invertibility is preserved in both directions.

The problem of Kaplansky on invertibility preservers is quite famous. And the matrix case is definitely the basic one. It is therefore somewhat surprising that the problem of the complete description of linear invertibility preserving maps on matrix algebras