Journal of Algebra 408 (2014) 147–165

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Automorphism groups of hyperbolic lattices

Michael H. Mertens [1,2]

Mathematisches Institut der Universität zu Köln, Weyertal 86-90, D-50931 Köln, Germany

ARTICLE INFO

Article history:
Received 29 January 2013
Available online 29 October 2013
Communicated by Gerhard Hiss

Keywords:
Hyperbolic/Lorentzian lattices
Automorphism group
Voronoi algorithm

ABSTRACT

Based on the concept of dual cones introduced by J. Opgenorth we give an algorithm to compute a generating system of the group of automorphisms of an integral lattice endowed with a hyperbolic bilinear form.

1. Introduction

In his famous paper *Nouvelles applications des paramètres continus à la théorie des formes quadratiques* [21], G.F. Voronoi presented an algorithm to enumerate (up to scaling) all perfect quadratic forms in a given dimension n. The general idea for that was to compute a face-to-face tessellation of a certain cone in the space of symmetric endomorphisms of $\mathbb{R}^n$ based on the pyramids induced by the shortest vectors of a perfect quadratic form.

Generalizing Voronoi's ideas, M. Koecher came up with the concept of *self-dual cones* or, as he called them, *positivity domains* [8,9] to obtain an alternative to the reduction theory of quadratic forms due to H. Minkowski.

Another slight generalization of these ideas to so called *dual cones* was then suiting for J. Opgenorth to find an algorithm to determine a generating system for the normalizer $N_{\mathrm{GL}_n(\mathbb{Z})}(G)$ of a finite unimodular group G of degree n (cf. [14]), which is an essential tool e.g. dealing with crystallographic space groups (cf. [15]).

E-mail address: mmertens@math.uni-koeln.de.
URL: http://www.mi.uni-koeln.de/~mmertens.

[1] The results in this paper are partially contained in the author's master's thesis [11] written under the supervision of Prof. Dr. Gabriele Nebe at Lehrstuhl D für Matematik, RWTH Aachen University, Templergraben 64, D-52062 Aachen, Germany.

[2] The author's research is supported by the DFG Graduiertenkolleg 1269 "Global Structures in Geometry and Analysis".

0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2013.09.034

The goal here is to use these methods to derive an algorithm which gives a generating system of the automorphism group of an integral hyperbolic lattice[3] (L, Φ) (see notation below).

There exists an algorithm due to E.B. Vinberg [20] to construct the maximal normal subgroup of the automorphism group of a hyperbolic lattice that is generated by reflections. But this algorithm terminates if and only if this reflection group has finite index in the full automorphism group. Lattices with this property, so-called *reflective* lattices, are very rare: For example if we consider the lattice $\mathbb{Z}^n$ together with the bilinear forms induced by matrices

$$H_n^{(d)} := diag(-d, 1, \dots, 1) \in \mathbb{Z}^{n\times n} \quad \text{for } d > 0,$$

then it is known that $(\mathbb{Z}^n, H_n^{(1)})$ is reflective if and only if $n \leqslant 19$ (cf. [20]). According to the classification of reflective hyperbolic lattices of rank 3 by D. Allock in [1], the highest prime divisor of the discriminant of such a lattice is 97 (cf. [1, p. 24]). Thus reflective lattices can neither occur in high dimensions nor for high discriminants.

To the author's knowledge so far there is no algorithm known to determine the automorphism group of a general hyperbolic lattice. The algorithm presented in this paper does at least not have theoretical limitations although in practice it can only handle lattices of small ranks and moderate discriminants (see Section 4).

The paper will be organized as follows: In Section 2 we recall the basic definitions and key results about dual cones from [14] which give a general method to determine generating systems of discontinuous groups acting on dual cones. The application of the results in Section 2 on hyperbolic lattices as well as a quite powerful way to shorten the calculation time is given in Section 3. Some examples and statistics are presented in Section 4.2. These were calculated using the computer algebra system MAGMA (cf. [3]). The source code for the necessary MAGMA-package `AutHyp.m` as well as a short description of the included intrinsics is available via the author's homepage http://www.mi.uni-koeln.de/~mmertens.

Notation. A *lattice* (L, Φ) always consists of two data, a free $\mathbb{Z}$-module L of rank n with basis $B = (b_1, \dots, b_n)$ and a non-degenerate, symmetric bilinear form $\Phi : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ where $\mathcal{V} = \mathbb{R} \otimes_{\mathbb{Z}} L$. The *signature* of the bilinear form will always be given as a pair $(p, -q)$ where p denotes the number of positive and q the number of negative eigenvalues of the *Gram matrix* of Φ with respect to B which is denoted by ${}_B\Phi^B := (\Phi(b_i, b_j))_{i,j=1}^n$. By $L^\#$ we denote the *dual lattice* of L and in case that L is integral, i.e. $L \subseteq L^\#$, we write $\Delta(L) := L^\#/L$ for the *discriminant group* of L. The *automorphism group* of a lattice (L, Φ) is defined as

$$\mathrm{Aut}(L) := \{ g \in \mathrm{GL}(\mathcal{V}) \mid Lg = L \text{ and } \Phi(xg, yg) = \Phi(x, y) \text{ for all } x, y \in L \}.$$

If (L, Φ) is an integral lattice, we can consider $\mathrm{Aut}(L)$ as a subgroup of $\mathrm{GL}_n(\mathbb{Z})$ by fixing a basis B of L:

$$\mathrm{Aut}(L) \cong \mathrm{Aut}_{\mathbb{Z}}(A) := \{ g \in \mathrm{GL}_n(\mathbb{Z}) \mid gAg^{tr} = A \},$$

where $A = {}_B\Phi^B$. Note that $\mathrm{Aut}(L)$ acts on L from the right while it acts from the left on the set of Gram matrices of L.

For any ring R let $R^n := R^{1\times n}$ be the free R-module of rank n represented as a row vector. By e_i we denote the ith row of the $n \times n$ unit matrix I_n.

For a subset S of any R-module M let $\langle S\rangle_R$ be the submodule of M generated by S (mostly we will omit the subscript R if there are no confusions about the base ring to be worried about). Similarly, if S is a subset of some group G, we denote by $\langle S\rangle$ the subgroup of G generated by S.

[3] In the literature often referred to as a Lorentzian lattice.