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The construction of finite solvable groups revisited

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We describe a new approach towards the systematic construction of finite groups up to isomorphism. This approach yields a practical algorithm for the construction of finite solvable groups up to isomorphism. We report on a GAP implementation of this method for finite solvable groups and exhibit some sample applications.

1. Introduction

The construction of all groups of a given order is an old and fundamental topic in finite group theory. Given an order n, the aim is to determine a list of groups of order n so that every group of order n is isomorphic to exactly one group in the list. There are many contributions to this topic in the literature. In the early history these are based on hand calculations; in more recent years algorithms have been developed for this purpose. We refer to [5] for a historic overview and a survey of the available algorithms.

Modern group construction algorithms distinguish three cases: nilpotent groups, solvable groups and non-solvable groups. Nilpotent groups are determined as direct products of p-groups and p-groups can be constructed using the p-group generation algorithm [21]. Solvable groups can be determined by the Frattini extension method [2] or the cyclic split extensions methods [3]. Non-solvable groups can be obtained via cyclic extensions of perfect groups as in [2] or via the method in [1].

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The p-group generation algorithm has been used to determine the groups of order dividing 2^9 [9] and the construction of the groups of order dividing p^7 for all primes p is also based on it, see [20] and [23]. The algorithm can also be used to determine groups with special properties; for example, it is a main tool in the investigation of finite p-groups by coclass, see [16] for background, and in the construction of restricted Burnside groups, see [19] and [22]. The p-group generation algorithm reduces the isomorphism problem to an orbit–stabilizer calculation.

The Frattini extension method and the cyclic split extension method have been used to determine most solvable non-nilpotent groups of order at most 2000, see [5]. The cyclic split extension method applies to groups with normal Sylow subgroup and order of the form $p^n \cdot q$ for different primes p and q only, while the Frattini extension method applies to all non-nilpotent solvable groups. The Frattini extension method uses a random isomorphism test for the reduction to isomorphism types.

The central aim of this paper is to introduce a new approach towards the systematic construction of groups up to isomorphism. This new approach is particularly useful for finite solvable groups and we developed it in detail and implemented it in GAP [28] for this case. For finite p-groups, our new approach coincides with the approach of p-group generation. In particular, the new approach reduces the solution of the isomorphism problem to an orbit–stabilizer calculation.

We discuss some sample applications of our new approach. We determined (again) all solvable non-nilpotent groups of order at most 2000 and thus checked the results available in the Small Groups library [4] and we constructed (for the first time) the groups of order $2304 = 3^2 \cdot 2^8$. We believe that our new approach could also be useful in the experimental investigation of coclass theory for finite solvable groups as suggested in [13] or in the construction of other finite solvable groups with special properties.

2. The general approach of the algorithm

In this section we exhibit a top-level introduction towards our new approach. The central idea is to use induction along a certain series: the so-called *F-central series*. We first introduce and investigate this series. Throughout this section, let G be a finite group.

Recall that the *Fitting subgroup* $F(G)$ is the maximal nilpotent normal subgroup of G. Define $\nu_0(G) = F(G)$ and let $\nu_{i+1}(G)$ be the smallest normal subgroup of $F(G)$ so that $\nu_i(G)/\nu_{i+1}(G)$ is a direct product of elementary abelian groups which is centralized by $F(G)$. Then we define the *F-central series* of G as

$$G \geqslant \nu_0(G) \geqslant \nu_1(G) \geqslant \cdots.$$

The following lemma provides an alternative characterization of the terms of the F-central series. We omit its straightforward proof. For an integer n with prime factorization $n = p_1^{e_1} \cdots p_r^{e_r}$ for different primes $p_1, \ldots, p_r$ and exponents $e_1, \ldots, e_r \neq 0$, we call $p_1 \cdots p_r$ the *core* of n.

Lemma 1. *Let G be a finite group and let k be the core of* $|F(G)|$. *Then*

$$\nu_{i+1}(G) = \big[F(G), \nu_i(G)\big]\nu_i(G)^k \quad \textit{for each } i \geqslant 0.$$

If $F(G)$ is a finite p-group, then the series $F(G) = \nu_0(G) \geqslant \nu_1(G) \geqslant \cdots$ coincides with the lower exponent-p central series of $F(G)$. In general, the group $F(G)$ is nilpotent and the series $F(G) = \nu_0(G) \geqslant \nu_1(G) \geqslant \cdots$ is a central series of $F(G)$. Thus there exists an integer c which $\nu_c(G) = \{1\}$. We call the smallest such integer the *F-class* of G. Further, if G has F-class at least 1, then the order of the quotient $\nu_0(G)/\nu_1(G)$ is called the *F-rank* of G. The next two lemmas collect some elementary facts about the F-central series. Let $\phi(G)$ denote the Frattini subgroup of G.

Lemma 2. *Let G be a finite group.*

(a) $\nu_1(G) = \phi(F(G)) \leqslant \phi(G)$.