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PBW-deformations of quantum groups

Yongjun Xu *, Shilin Yang [1]

College of Applied Sciences, Beijing University of Technology, Beijing 100124, China

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ABSTRACT

In this paper, we investigate certain deformations $\mathfrak{B}_q(\mathfrak{g})$ of the negative part $U_q^-(\mathfrak{g})$ of quantized enveloping algebras $U_q(\mathfrak{g})$. An algorithm is established to determine when a given $\mathfrak{B}_q(\mathfrak{g})$ is a PBW-deformation of $U_q^-(\mathfrak{g})$. For $\mathfrak{g}$ of type $\mathbb{A}_2$ and $\mathbb{B}_2$, we classify PBW-deformations of $U_q^-(\mathfrak{g})$. Moreover, we explicitly construct some PBW bases for a class of PBW-deformations $\mathfrak{B}_q(\mathfrak{g})$ of $U_q^-(\mathfrak{g})$. As an application, Iorgov–Klimyk's PBW bases for the non-standard quantum deformation $U_q'(\mathfrak{so}(n,\mathbb{C}))$ of the universal enveloping algebra $U(\mathfrak{so}(n,\mathbb{C}))$ are recovered.

1. Introduction

A filtered algebra $\mathbb{U}$ is called a Poincaré–Birkhoff–Witt-deformation (abbr. PBW-deformation) of a graded algebra $\mathcal{A}$ if its associated graded algebra $\mathrm{gr}(\mathbb{U})$ is isomorphic to $\mathcal{A}$. PBW-deformation theory of graded algebras is extensively studied. For Koszul or $\mathbb{N}$-Koszul algebras, a Jacobi type condition was given for the determination of PBW-deformation (see e.g. [2,4,10,33]). While for an arbitrary graded algebra $\mathcal{A}$ over a field, Cassidy and Shelton [6] extended the above results to a more general Jacobi condition for deciding when certain deformations of $\mathcal{A}$ obtained by altering its defining relations are PBW ones.

For all complex simple Lie algebras $\mathfrak{g}$, Drinfeld [9] and Jimbo [20] introduced the quantum deformations $U_q(\mathfrak{g})$ of universal enveloping algebras $U(\mathfrak{g})$, which are very important in mathematical

* Corresponding author.
E-mail address: yjxu2002@163.com (Y. Xu).

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physics (see also [19,21]). Except the cases $U_q^-(\mathfrak{sl}_2)$ and $U_q^-(\mathfrak{sl}_3)$, the negative nilpotent subalgebras $U_q^-(\mathfrak{g})$ of $U_q(\mathfrak{g})$ are graded algebras with defining relations in mixed degrees (the word 'mixed' means that there exist at least two defining relations whose degrees are different, cf. [6]). Some PBW-deformations of $U_q^-(\mathfrak{g})$ appeared in the investigation of coideal subalgebras of $U_q(\mathfrak{g})$ (cf. [24–27,31, 32]) and non-standard quantum deformations of $U(\mathfrak{g})$ (cf. [12,17]). In the theory of quantum groups, Lusztig [28,29] investigated braid group actions on $U_q(\mathfrak{g})$ which allow the definition of root vectors and PBW bases. In analogy to the quantum group case, braid group actions on coideal subalgebras of $U_q(\mathfrak{g})$ were also investigated by many authors (see e.g. [7,22,30]).

Assume that $\mathfrak{g}$ is a complex simple Lie algebra of rank n. Let $A=(a_{ij})_{n\times n}$ be the Cartan matrix of $\mathfrak{g}$ and $H=(h(B_i,B_j))_{n\times n}$ with $h(B_i,B_j)$ in the free algebra generated by $B_1,B_2,\dots,B_n$ and $\mathrm{Deg}(h(B_i,B_j))<2-a_{ij}$. Denote by $\mathfrak{B}_q(\mathfrak{g})$ the quantum algebra with generators B_i $(1\leqslant i\leqslant n)$ and defining relations

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\begin{pmatrix}1-a_{ij}\\ k\end{pmatrix}_i B_i^{1-a_{ij}-k}B_jB_i^k=h(B_i,B_j),\quad 1\leqslant i\neq j\leqslant n.$$

Motivated by the above mentioned research, in this present paper we mainly focus on the following two problems:

(1) Determining when a given deformation $\mathfrak{B}_q(\mathfrak{g})$ of $U_q^-(\mathfrak{g})$ is a PBW-deformation.
(2) Constructing PBW bases for a class of PBW-deformations $\mathfrak{B}_q(\mathfrak{g})$ of $U_q^-(\mathfrak{g})$.

For the problem (1), our main techniques are the Jacobi condition given in [6] and the Bernstein–Gelfand–Gelfand resolution (abbr. BGG-resolution) established in [14]. The Jacobi condition in [6] actually transforms the problem of determining PBW-deformations of a graded algebra $\mathcal{A}$ into a series of linear algebra problems which we denote $(*)$. Though it is a sufficient and necessary condition for judging which deformations $\mathbb{U}$ of $\mathcal{A}$ are PBW ones, there is a homological constant $c(\mathcal{A})$ in it whose accurate value is generally not easy to obtain. In [6], $c(\mathcal{A})$ is called the complexity of $\mathcal{A}$ which in a sense reflects the scale of the sets of linear equations in $(*)$. By Definition 2.1, the size of $c(\mathcal{A})$ is deeply related with the bigraded Yoneda algebra $E(\mathcal{A})=\bigoplus \mathrm{Ext}_{\mathcal{A}}^{r,s}(\mathbb{C},\mathbb{C})$ of $\mathcal{A}$. For finite dimensional semisimple Lie algebras the BGG-resolution was introduced in [3]. The quantum group version of the BGG-resolution was established in [14] and explicitly written down in [13]. In this paper, we compute the complexity $c(U_q^-(\mathfrak{g}))$ of $U_q^-(\mathfrak{g})$ by using the BGG-resolution of the trivial left $U_q^-(\mathfrak{g})$-module ${}_{U_q^-(\mathfrak{g})}\mathbb{C}$. Based on the above ideas, we propose an algorithm to decide if a given algebra $\mathfrak{B}_q(\mathfrak{g})$ is a PBW-deformation of $U_q^-(\mathfrak{g})$. In practical use, our algorithm is very technical because the amount of calculations in it is very large for hand computation. So the computer program realization of our algorithm or more conceptional research on the classification of PBW-deformations of $U_q^-(\mathfrak{g})$ is interesting.

The algebras $\mathfrak{B}_q(\mathfrak{g})$ in problem (2) can be viewed as a uniform description of some coideal subalgebras of $U_q(\mathfrak{g})$ in [22] and Iorgov–Klimyk's non-standard quantum deformation $U_q'(\mathfrak{so}(n,\mathbb{C}))$ in [12]. In fact, they were studied by Letzter in more generality in [24–27], and proved to be coideal subalgebras of $U_q(\mathfrak{g})$ and PBW-deformations of $U_q^-(\mathfrak{g})$ in [24]. Our results indicate that Kolb–Pellegrini's braid group actions on $\mathfrak{B}_q(\mathfrak{g})$ also allow the definition of root vectors and some PBW bases $B(w_0)$ for $\mathfrak{B}_q(\mathfrak{g})$. The root vectors of $\mathfrak{B}_q(\mathfrak{g})$ have the same form as those of $U_q(\mathfrak{g})$ for simply laced $\mathfrak{g}$, while for non-simply laced $\mathfrak{g}$ they have some additional terms of lower degree. In our proof of the PBW theorems for $\mathfrak{B}_q(\mathfrak{g})$, it is crucial that $\mathfrak{B}_q(\mathfrak{g})$ are coideal subalgebras of $U_q(\mathfrak{g})$ and PBW-deformations of $U_q^-(\mathfrak{g})$.

This paper is organized as follows. In Section 2, we fix some notations and collect the background material that will be necessary in the sequel. In Section 3, we give an algorithm for problem (1) after theoretical analysis, then apply it to the case $\mathfrak{g}$ of type $\mathbb{A}_2$ and $\mathbb{B}_2$. In Section 4 we explicitly construct some PBW bases $B(w_0)$ for $\mathfrak{B}_q(\mathfrak{g})$ in problem (2). In Section 4.1, we state some properties of the algebra automorphisms τ_i $(1\leqslant i\leqslant n)$ of $\mathfrak{B}_q(\mathfrak{g})$ given by Kolb and Pellegrini, then calculate some