Journal of Algebra 409 (2014) 148–161

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

The extended affine Lie algebra associated with a connected non-negative unit form

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ARTICLE INFO

Article history:
Received 27 August 2013
Available online 18 April 2014
Communicated by Nicolás Andruskiewitsch

MSC:
17B67

Keywords:
Extended affine Lie algebra
Extended affine root system
Quadratic form
Kac–Moody algebra

ABSTRACT

Given a connected non-negative unit form we construct an extended affine Lie algebra by giving a Chevalley basis for it. We also obtain this algebra as a quotient of an algebra defined by means of generalized Serre relations by M. Barot, D. Kussin and H. Lenzing. This is done in an analogous way to the construction of the simply-laced affine Kac–Moody algebras. Thus, we obtain a family of extended affine Lie algebras of simply-laced Dynkin type and arbitrary nullity. Furthermore, there is a one-to-one correspondence between these Lie algebras and the equivalence classes of connected non-negative unit forms.

1. Introduction

To each unit form $q : \mathbb{Z}^n \to \mathbb{Z}$, M. Barot, D. Kussin and H. Lenzing associate in [3] a complex Lie algebra $\tilde{G}(q)$ by means of generalized Serre relations. We are interested in the case when the form q is connected and non-negative, see Section 2 for definitions. When the rank of the radical of q is 0 or 1, the algebra $\tilde{G}(q)$

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[1] The author wishes to acknowledge the support of M. Barot, who directed the author's undergraduate thesis [6] from which this article is derived. This acknowledgment is extended to L. Demonet for his helpful comments and suggestions on an earlier version of this article which greatly improved its readability.

http://dx.doi.org/10.1016/j.jalgebra.2014.03.029
0021-8693/

is a simply-laced finite dimensional simple Lie algebra or a simply-laced affine Kac–Moody algebra respectively. Also, the algebra $\tilde{G}(q)$ admits a root space decomposition with respect to a finite dimensional abelian Cartan subalgebra $\tilde{H}$ with the roots of q, *i.e.* the set $R(q) := q^{-1}(0) \cup q^{-1}(1)$, as root system. Moreover, $R(q)$ is a so-called extended affine root system, see Definition 2.2 and Proposition 2.4. Extended affine root systems are a certain generalization of affine root systems; one important difference is that, in general, isotropic roots generate a subspace (of a Euclidean space, say) of dimension higher than 1. Extended affine root systems are precisely the root systems associated with extended affine Lie algebras. Roughly speaking, extend affine Lie algebras are characterized by the existence of a non-degenerate symmetric invariant bilinear form which induces a root-space decomposition with respect to a finite dimensional abelian subalgebra; moreover, the adjoint representations of homogeneous elements of non-isotropic degree are locally nilpotent, see Definition 2.1. Examples of extended affine Lie algebras are both the simple finite-dimensional Lie algebras and the affine Kac–Moody algebras. We refer the reader to [1] for an introductory treatment of the theory of extended affine Lie algebras and motivation for their study.

Our aim is to study the connection between extended affine Lie algebras and the Lie algebra $\tilde{G}(q)$ described above. In fact, the Lie algebra $\tilde{G}(q)$ is very close to being an extended affine Lie algebra, but in general it lacks an invariant non-degenerate symmetric bilinear form (the non-degeneracy being the main obstruction). We can obtain such a form by passing to a natural quotient of $\tilde{G}(q)$. Thus, this article can be regarded as a simple extension of [3] towards the theory of extended affine Lie algebras.

Let us explain the contents of this article. In Section 2, we recall the definitions of an extended affine Lie algebra and of an extended affine root system. We also show how to associate an extended affine root system to a given connected non-negative unit form. In Section 3, given a connected non-negative unit form q, we construct a Lie algebra $E(q)$ with root system $R(q)$ and with nullity the corank of q. This construction is a slight modification of the construction of [3, Sec. 2], which also is related to a construction given by Borcherds in [4]. The modification concerns both the Cartan subalgebra and the root spaces associated to non-isotropic roots. Our main result is the following:

Theorem 1.1. *(See Theorems 3.7 and 4.5.) Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a connected non-negative unit form with associated root system $R(q)$. Then the Lie algebra $E(q)$ is an extended affine Lie algebra with root system $R(q)$. Furthermore, let q and q' be connected non-negative unit forms. Then, q and q' are equivalent if and only if $E(q)$ and $E(q')$ are isomorphic as graded Lie algebras.*

In Section 4, we recall from [3] the construction of the Lie algebra $\tilde{G}(q)$ by means of generalized Serre relations. We show that $E(q)$ can also be obtained from $\tilde{G}(q)$ in a way that imitates the construction of the affine Kac–Moody algebras, *cf.* [5, Chapter 14]. More precisely, we have the following result: