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Rings and modules characterized by opposites of injectivity

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ABSTRACT

In a recent paper, Aydoğdu and López-Permouth have defined a module M to be N-subinjective if every homomorphism $N \to M$ extends to some $E(N) \to M$, where $E(N)$ is the injective hull of N. Clearly, every module is subinjective relative to any injective module. Their work raises the following question: What is the structure of a ring over which every module is injective or subinjective relative only to the smallest possible family of modules, namely injectives? We show, using a dual opposite injectivity condition, that such a ring R is isomorphic to the direct product of a semisimple Artinian ring and an indecomposable ring which is (i) a hereditary Artinian serial ring with $J^2 = 0$; or (ii) a QF-ring isomorphic to a matrix ring over a local ring. Each case is viable and, conversely, (i) is sufficient for the said property, and a partial converse is proved for a ring satisfying (ii). Using the above mentioned classification, it is also shown that such rings coincide with the fully saturated rings of Trlifaj except, possibly, when von Neumann regularity is assumed. Furthermore, rings and abelian groups which satisfy these opposite injectivity conditions are characterized.

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1. Introduction and preliminaries

There is an extensive literature on the injectivity of modules and rings, a considerable amount of which involves notions derived from relative injectivity, i.e. that of a module with respect to a fixed module, inspired by Baer's criterion. Introduced in a similar vein, the kind of injectivity (namely, subinjectivity) and the opposite of injectivity induced by it (namely, indigence), as discussed by Aydoğdu and López-Permouth in [3], seem to offer a new perspective on the topic and to be abundant with interesting questions: Let R be an associative ring with identity and M and N be unital right R-modules. M is said to be N-subinjective if every homomorphism $N \to M$ can be extended to some homomorphism $E(N) \to M$, where $E(N)$ is the injective hull of N. The class of all modules N such that M is N-subinjective is called the subinjectivity domain of M, and is denoted by $\underline{In}^{-1}(M)$. If N is injective, then M is vacuously N-subinjective. So, the smallest possible subinjectivity domain is the class of injective modules. A module with such a subinjectivity domain is defined in [3] to be indigent. The existence of indigent modules for an arbitrary ring is unknown, but an affirmative answer is known for some rings such as $\mathbb{Z}$ and Artinian serial rings (see [3]).

In this paper, we address some questions raised by and studied in [3]. The first question considered here is the following: What is the structure of a ring over which every right module is indigent or injective? In order to approach this problem, we use the following notion, which is a sort of dual to the notion of indigence: We call a module M a test for injectivity by subinjectivity (t.i.b.s.) if the only modules which are M-subinjective are the injective ones. Such modules exist over any ring (Proposition 1). An easy observation shows that the rings in question are precisely those whose modules are injective or t.i.b.s. (Proposition 2). We then prove that such a ring is isomorphic to the direct product of a semisimple Artinian ring and an indecomposable ring T such that (i) T is a hereditary Artinian serial ring with $J^2 = 0$; or, (ii) T is a QF-ring isomorphic to a matrix ring over a local ring (Theorem 3). An example of each case exists. Conversely, the condition (i) is a sufficient one for an indecomposable ring (Proposition 10), and we have a partial converse for the case (ii), which yields a characterization of QF-rings that are isomorphic to a matrix ring over a local ring (Theorem 14).

Next, we mention the connection between rings whose right modules are injective or indigent and fully saturated rings of Trlifaj [14]. Using Theorem 3 (see the preceding paragraph), it is shown that the two types of rings coincide when the underlying ring is not semisimple or a von Neumann regular ring (Theorem 16). It follows as a consequence that nonsemisimple rings of the former type is in fact a subclass of the latter (Corollary 17).