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Invariants, exponents and formal group laws

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A B S T R A C T

Let W be the Weyl group of a crystallographic root system acting on the associated weight lattice by means of reflections. In the present note we extend the notion of exponent of the W-action to the context of an arbitrary algebraic oriented cohomology theory of Levine–Morel and Panin–Smirnov and the associated formal group law. From this point of view the classical Dynkin index of the associated Lie algebra will be the second exponent of the deformation map from the multiplicative to the additive formal group law. We apply this generalized exponent to study the torsion part of an arbitrary oriented cohomology theory of a twisted flag variety.

1. Introduction

Let W be the Weyl group of a crystallographic root system which acts by means of simple reflections on the respective weight lattice Λ. Consider the induced actions of W on the polynomial ring $S^*(\Lambda)$ and the integral group ring $\mathbb{Z}[\Lambda]$ of Λ. The ring of invariants $\mathbb{Z}[\Lambda]^W$ can be identified with the representation ring of the associated linear algebraic group G and according to the celebrated Chevalley theorem is a polynomial ring in classes of *fundamental representations*. On the other hand $S^*(\Lambda)^W \otimes \mathbb{Q}$ is known to be a polynomial ring as well with generators given by *basic polynomial invariants*.

The main goal of paper [1] was to establish the relationship between these two sets of invariants by means of the *Chern class map*. This was done by introducing the notion of an *exponent* – an integer τ_d which measures the difference between these invariants. In particular, it was proven that τ_2 coincides with the *Dynkin index* of the associated Lie

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algebra (see Theorem 4.4 in [1]). Using the *Riemann–Roch* theorem and the Grothendieck γ-*filtration* it was also shown [1, Corollary 6.8] that the exponent τ_d bounds the annihilator of the torsion part of the Chow groups CH^d for $d = 2, 3, 4$ of some twisted flag varieties, hence, providing new estimates for the torsion in small codimensions.

The next step was done in [2] where it was shown that τ_d divides the Dynkin index τ_2 for all $d \geq 2$. This fact together with Demazure's description of the kernel of the characteristic map allowed to obtain a uniform bound for the annihilator of the torsion of CH^d of strongly inner forms of flag varieties for all d, hence, pushing geometric applications of the exponent even further.

The goals of the present paper are

a) to extend the notion of an exponent to the context of arbitrary algebraic *oriented cohomology theories* (o.c.t.) and the associated *formal group laws* (f.g.l.). From this point of view, the Dynkin index of a root system will be just the second exponent of the deformation from the multiplicative to the additive f.g.l.;
b) to apply these generalized exponents (as in [1] and [2]) to estimate the torsion part of an arbitrary algebraic o.c.t. of some flag varieties.

Recall that the notion of an algebraic o.c.t. was introduced by Levine and Morel [10] and by Panin and Smirnov [12]. Roughly speaking, it is a cohomological-type functor $\mathtt{h}$ from the category of smooth varieties over a field to the category of commutative rings endowed with push-forward maps and characteristic classes (see Section 6). Basic examples of such functors are the Chow ring of algebraic cycles modulo the rational equivalence relation, the Grothendieck K_0, the algebraic cobordism Ω of Levine and Morel (see [12, §2.1, 2.5, 3.8] for more examples).

The theory of formal group laws originated from the theory of Lie groups. Roughly speaking, an f.g.l. is a formal power series which behaves as if it was the product of a Lie group. Its subclass of one-dimensional commutative formal group laws had been intensively used in topology, especially in cobordism theory and, more generally, for studying complex oriented theories. A link between oriented cohomology and formal group laws is given by the Quillen formula expressing the first characteristic class $\mathrm{c}_1^{\mathtt{h}}$ of a tensor product of two line bundles $\mathrm{c}_1^{\mathtt{h}}(L_1 \otimes L_2) = F(\mathrm{c}_1^{\mathtt{h}}(L_1), \mathrm{c}_1^{\mathtt{h}}(L_2))$, where F is the one-dimensional commutative f.g.l. over the coefficient ring R associated to $\mathtt{h}$.

We then define the key object of the present paper as follows: According to [3] the algebras $S^*(\Lambda)$ and $\mathbb{Z}[\Lambda]$ can be viewed as specializations (for additive and multiplicative F resp.) of a more general object – the formal group algebra $R[[\Lambda]]_F$. Consider its subring of invariants $R[[\Lambda]]_F^W$ and the associated ideal $\mathcal{I}_F^W$. Given two formal group laws F and F' there is a (non-canonical and non-W-equivariant) isomorphism $\Phi: R[[\Lambda]]_F \xrightarrow{\simeq} R[[\Lambda]]_{F'}$ called the *deformation map*. The (generalized) *exponent* $\tau_d^{F,F'}$ from F to F' is defined to be the exponent of the image $\Phi(\mathcal{I}_F^W)$ in $\mathcal{I}_{F'}^W$ after passing to the d-th subsequent quotients of the $\mathcal{I}_F$- and $\mathcal{I}_{F'}$-adic filtrations (see Definition 5.1). In case F is the multiplicative and F' is the additive f.g.l. we obtain the exponent defined in [1].