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A Strong Abhyankar–Moh Theorem and Criterion of Embedded Line

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ABSTRACT

The condition of plane polynomial curve to be a line in well-known Abhyankar–Moh Theorem is replaced by weaker ones. A Criterion of Embedded Line is obtained from this strong theorem: Two polynomials can generate the entire polynomial ring iff their derivatives can be generated.

1. Introduction

Famous Abhyankar–Moh Theorem [1,2] states that for a field k of characteristic zero, if $f(z)$ and $g(z)$ are polynomials and the polynomial ring $k[f(z), g(z)] = k[z]$, then either $\deg f(z)$ divides $\deg g(z)$ or $\deg g(z)$ divides $\deg f(z)$. But to require the considered polynomial curve to be a line at beginning is too strong and limits the applications of the theorem. We find that as long as $\deg f(z) - c$ and $\deg g(z) - c$ are in the degree semigroup

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of polynomial ring $k[f(z), g(z)]$, where positive integer $c \leq \min(\deg f(z), \deg g(z))$, we have that either $\deg f(z)$ divides $\deg g(z)$ or $\deg g(z)$ divides $\deg f(z)$. Therefore we call it Strong Abhyankar–Moh Theorem. Using this strong theorem, we get a criterion for a polynomial plane curve to be an embedded line.

2. Planar semigroups

Following [5], we first define characteristic δ-sequence and planar semigroup.

Definition 2.1. Let $\delta = (\delta_0, \delta_1, \cdots, \delta_h)$ $(h \geq 1)$ be a sequence of $h+1$ natural numbers. And let $d_k = \gcd(\delta_0, \delta_1, \cdots, \delta_{k-1})$ $(1 \leq k \leq h+1)$. Then δ is called a characteristic δ-sequence and the semigroup $\Gamma = \Gamma(\delta)$ is called a planar semigroup if the following conditions are satisfied:

(1) $d_1 \geq d_2 > d_3 > \cdots > d_h > d_{h+1} = 1$,
(2) $\delta_k \frac{d_k}{d_{k+1}} \in \Gamma(\delta_0, \delta_1, \cdots, \delta_{k-1})$ $(1 \leq k \leq h)$,
(3) $\delta_k < \delta_{k-1} \frac{d_{k-1}}{d_k}$ $(2 \leq k \leq h)$.

The following concept of standard expansion is also from [5].

Definition 2.2. Let $\delta = (\delta_0, \delta_1, \cdots, \delta_h)$ $(h \geq 1)$ be a characteristic δ-sequence and let s be an integer. If

$$s = a_0\delta_0 + a_1\delta_1 + \cdots + a_h\delta_h \quad \text{where } 0 \leq a_i < d_i/d_{i+1} \ (1 \leq i \leq h) \tag{2.2.1}$$

then we say that s has standard expansion with respect to δ.

The following properties of characteristic δ-sequence are well-known:

Lemma 2.3. *(See [5].) Let $\delta = (\delta_0, \delta_1, \cdots, \delta_h)$ $(h \geq 1)$ be a characteristic δ-sequence. Then for any integer s, there is unique standard expansion (2.2.1). Moreover,*

(1) *$s \in \Gamma(\delta)$ iff $a_0 \geq 0$.*
(2) *If $d_i | s$, then $a_j = 0$ for $i \leq j \leq h$.*

We consider standard expansions of two integers and generalize (2) of the above Lemma 2.3.

Proposition 2.4. *Let $\delta = (\delta_0, \delta_1, \cdots, \delta_h)$ $(h \geq 1)$ be a characteristic δ-sequence.*

(1) *Let s_1 and s_2 be two integers with standard expansions*

$$s_k = a_{k0}\delta_0 + a_{k1}\delta_1 + \cdots + a_{kh}\delta_h \quad \textit{where } 0 \leq a_{ki} < d_i/d_{i+1} \ (1 \leq i \leq h) \tag{2.4.1}$$