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Stanley depths of certain Stanley–Reisner rings

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Let Δ be a simplicial complex on $[n]$, $S = K[x_1, \ldots, x_n]$ the polynomial ring in n-variables over a field K and $K[\Delta] = S/I_\Delta$ the Stanley–Reisner ring of Δ with respect to K. It is proved that the Stanley Conjecture holds for $K[\Delta]$, i.e., $\mathrm{sdepth}_S(K[\Delta]) \geq \mathrm{depth}_S(K[\Delta])$, when I_Δ has four associated prime ideals. It is also obtained that $\mathrm{sdepth}_S(K[\Delta]) \geq \mathrm{size}_S(I_\Delta)$.

1. Introduction

Let K be a field and $S = K[x_1, \ldots, x_n]$ the polynomial ring in n-variables over K. Suppose that Δ is a simplicial complex on $[n]$. Let I_Δ be the Stanley–Reisner ideal of Δ over K. Then I_Δ is a square-free monomial ideal of S and conversely, every square-free monomial ideal of S is the Stanley–Reisner ideal of some simplicial complex on $[n]$ over K. The Stanley–Reisner ring $K[\Delta]$ of Δ with respect to K is the ring S/I_Δ, which is a cyclic S-module.

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Let I be a square-free monomial ideal of S. Then $I = P_1 \cap \cdots \cap P_s$ where the prime ideals P_k, $k = 1, \ldots, s$, have the form $(x_{i_1}, \ldots, x_{i_t})$ and are not included one in other. Note that $\{P_1, \ldots, P_s\}$ is just the set of associated prime ideals of I.

Let M be a finitely generated $\mathbb{Z}^n$-graded S-module. Then a Stanley decomposition $\mathcal{D}$ of M is a finite direct sum of K-spaces:

$$\mathcal{D} : M = \bigoplus_{i=1}^{r} m_i K[Z_i],$$

where $m_i \in M$ is homogeneous and $Z_i \subseteq \{x_1, \ldots, x_n\}$, $i = 1, \ldots, r$, and its Stanley depth, $\operatorname{sdepth}_S(\mathcal{D})$, is defined as $\min\{|Z_i| \mid i = 1, \ldots, r\}$. The Stanley depth of M is, by definition, the following

$$\operatorname{sdepth}_S(M) = \max\bigl\{\operatorname{sdepth}_S(\mathcal{D}) \bigm| \mathcal{D} \text{ is a Stanley decomposition of } M\bigr\}.$$

Stanley [7] conjectured that $\operatorname{sdepth}_S(M) \geq \operatorname{depth}_S(M)$. There are many researches on this conjecture, especially when M has the form S/I or I with I a square-free monomial ideal of S. Popescu and Qureshi [6] showed that the Stanley Conjecture holds for S/I when I is an intersection of two or three monomial prime ideals, and for I when I is an intersection of two monomial prime ideals. It was proved in [4] that the Stanley Conjecture remains true for I when I is an intersection of three monomial prime ideals. Recently, Popescu [5] showed that the Stanley Conjecture holds for any square-free monomial ideal I of S which is an intersection of four monomial prime ideals. These two results on I depend on one kind of decomposition of I. By using our decomposition of S/I, we will show that the Stanley Conjecture holds for S/I where I is an intersection of four monomial prime ideals.

The size of a monomial ideal was introduced by Lyubeznik [3]. It was proved there a well-known result that $\operatorname{depth}_S(I) \geq 1+\operatorname{size}_S(I)$. For the Stanley depth, Herzog, Popescu and Vladoiu [1] obtained a similar result that $\operatorname{sdepth}_S(I) \geq 1+\operatorname{size}_S(I)$ for any monomial ideal I. In [1], they expected that $\operatorname{sdepth}_S(S/I) \geq \operatorname{size}_S(I)$ holds too. We will show that it is true for any square-free monomial ideal by using our decomposition of S/I.

2. Stanley decompositions

Let I be a square-free monomial ideal of $S = K[x_1, \ldots, x_n]$. According to one decomposition of I as an intersection of monomial prime ideals, we obtain one decomposition of the ring S. This decomposition is similar to those from [4,5,1] and the proof of the next proposition is taken from [5, Theorem 2.6].

Proposition 2.1. *Let $S' = K[x_1, \ldots, x_r]$, $S'' = K[x_{r+1}, \ldots, x_n]$, $S = K[x_1, \ldots, x_n]$, and I be a square-free monomial ideal of S which is an intersection of monomial prime ideals:*

$$I = P_1 \cap \cdots \cap P_s, s \geq 2.$$