Journal of Algebra 403 (2014) 77–92

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Local rings with zero-dimensional formal fibers[☆]

Đoàn Trung Cường[1]

Institute of Mathematics, VAST, 18 Hoang Quoc Viet, 10307 Hanoi, Vietnam[2]

ARTICLE INFO

Article history:
Received 13 September 2013
Available online 25 January 2014
Communicated by Kazuhiko Kurano

MSC:
13B35
14B05

Keywords:
Dimension of formal fiber
Local ring with trivial formal fibers
Weierstrass Preparation Theorem
Lichtenbaum–Hartshorne Theorem

ABSTRACT

We study Noetherian local rings whose all formal fibers are of dimension zero. Universal catenarity and going-up property of the canonical map to the completion are considered. We present several characterizations of these rings, including a characterization of Weierstrass preparation type. A characterization of local rings with going up property by a strong form of Lichtenbaum–Hartshorne Theorem is obtained. As an application, we give an upper bound for dimension of formal fibers of a large class of algebras over these rings.

0. Introduction

Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring. In [11], Matsumura proposes to study the maximum of the dimensions of all formal fibers of R. Denote this number by $\alpha(R)$. In an attempt to relate this invariant with other invariants of R, Matsumura gives several estimates of $\alpha(R)$ in terms of the dimension of R and computes it for some concrete examples. In general one has $0 \leqslant \alpha(R) \leqslant \dim(R) - 1$. Matsumura constructs

☆ This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.01-2012.05.

E-mail address: doantc@gmail.com.

1 Current address: Vietnam Institute for Advanced Study in Mathematics, Ta Quang Buu Building, 01 Dai Co Viet, Hai Ba Trung, Hanoi, Vietnam.

2 Permanent address.

http://dx.doi.org/10.1016/j.jalgebra.2014.01.002

examples of local rings with $\alpha(R) = 0, \dim(R) - 1, \dim(R) - 2$ (cf. [11]). Later, in an attempt to answer a question of Matsumura, Rotthaus [16] constructs examples with $\alpha(R) = i$ for any $i \in \{0, 1, \ldots, \dim(R) - 1\}$. However, the interesting question how to compute $\alpha(R)$ by means of known numerical invariants of R is not yet answered. To find an answer to this question, beginning with understanding the class of local rings whose formal fibers have dimension zero seems to be natural. These local rings will be called local rings with trivial formal fibers. They are the object of investigation in this note.

Examples of local rings with trivial formal fibers consist of complete local rings and one-dimensional local rings. Higher dimensional and non-complete examples are harder to construct. A well-known example is due to Nagata [13, Example E3.1, Appendix]. There is also a construction of local rings with trivial formal fibers due to Rotthaus [16]. There are interesting examples coming from arithmetics and geometry. For instance, let K be a p-adic field, that is, a finite extension of $\mathbb{Q}_p$, and let X be a projective curve over the ring of integers $\mathcal{O}_K$. If x is a point of the special fiber of X, then the local ring $\mathcal{O}_{X,x}$ of germs of regular functions at x has all formal fibers of dimension zero (see Example 1.1). This is in fact very special class of local rings with trivial formal fibers which is a motivation for our present work. Further examples with applications in algebraic number theory could be found in [6,3].

Local rings with trivial formal fibers have been studied by several authors. In [7], Heinzer and Rotthaus consider excellent Henselian local rings with trivial formal fibers and show that they satisfy the Noetherian intermediate rings property. Recently, Zöschinger [17] gave several characterizations of local rings whose canonical map to the completion satisfies the going-up property. These are in fact local rings with trivial formal fibers.

The aim of this note is to study several basic properties of local rings with trivial formal fibers and some applications to higher dimensional case.

In Section 1 we first present the example of local rings of regular functions on a curve over the ring of p-adic integers, these rings have trivial formal fibers. The main result of this section is Theorem 1.4 and its consequence where we consider over a local ring with trivial formal fibers certain class of algebras, including the polynomial algebras, and give an upper bound for the dimension of their formal fibers.

Universally catenary local rings with trivial formal fibers are considered in Section 2. By using a result of Charters and Loepp, we obtain an example of a local ring with trivial formal fibers which is not universally catenary. This distinguishes the subclass to the whole class of local rings considered in the previous section. We then present several characterizations of local rings in this subclass including a characterization of Weierstrass preparation type (Theorem 2.4 and its consequence).

In the last Section 3 we consider local rings whose canonical map $R \hookrightarrow \widehat{R}$ satisfies the going-up theorem. An easy argument shows that R is universally catenary and $\alpha(R) = 0$. The converse does not hold (see Example 3.1). Recently Zöschinger [17] studied such rings with going-up property and gave several characterizations. We will give another characterization by means of a strong form of Lichtenbaum–Hartshorne Vanishing Theorem.