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Conjugation in semigroups

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ABSTRACT

The action of any group on itself by conjugation and the corresponding conjugacy relation play an important role in group theory. There have been several attempts to extend the notion of conjugacy to semigroups. In this paper, we present a new definition of conjugacy that can be applied to an arbitrary semigroup and it does not reduce to the universal relation in semigroups with a zero. We compare the new notion of conjugacy with existing definitions, characterize the conjugacy in various semigroups of transformations on a set, and count the number of conjugacy classes in these semigroups when the set is infinite.

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1. Introduction

Let G be a group. For elements $a, b \in G$, we say that a is *conjugate* to b if there exists $g \in G$ such that $b = g^{-1}ag$. It is clear that this relation is an equivalence on G and that a is conjugate to b if and only if there exists $g \in G$ such that $ag = gb$. Using the latter formulation, one may try to extend the notion of conjugacy to semigroups in the following way: define a relation $\sim_l$ on a semigroup S by

$$a \sim_l b \quad \Leftrightarrow \quad \exists_{g \in S^1}\ ag = gb, \tag{1.1}$$

where S^1 is S with an identity adjoined. If $a \sim_l b$, we say that a is *left conjugate* to b [34,39,40]. (We will write "$\sim$" with various subscripts for possible definitions of conjugacy in semigroups. The subscript in $\sim_l$ comes from the name "left conjugate.") In a general semigroup S, the relation $\sim_l$ is reflexive and transitive, but not symmetric. If S has a zero, then $\sim_l$ is the universal relation $S \times S$. The relation $\sim_l$ is an equivalence in any free semigroup. Lallement [28] has defined the conjugate elements of a free semigroup S as those related by $\sim_l$ and showed that $\sim_l$ is equal to the following equivalence on the free semigroup S:

$$a \sim_p b \quad \Leftrightarrow \quad \exists_{u,v \in S^1}\ a = uv \text{ and } b = vu. \tag{1.2}$$

In a general semigroup S, the relation $\sim_p$ is reflexive and symmetric, but not transitive. If $a \sim_p b$ in a general semigroup, we say that a and b are *primarily conjugate* [27] (hence the subscript in $\sim_p$). Kudryavtseva and Mazorchuk [26,27] have defined the transitive closure $\sim_p^*$ of $\sim_p$ as a conjugacy relation in a general semigroup. (See also [18].)

Otto [34] has studied the relations $\sim_l$ and $\sim_p$ in the monoids S presented by finite Thue systems, and introduced a new definition of conjugate elements in such an S:

$$a \sim_o b \quad \Leftrightarrow \quad \exists_{g,h \in S^1}\ ag = gb \text{ and } bh = ha. \tag{1.3}$$

(Since S is a monoid, $S^1 = S$. However, we wanted to write the definition of $\sim_o$ so that it would apply to any semigroup.) For any semigroup S, $\sim_o$ is an equivalence on S, and so it provides another possible definition of conjugacy in a general semigroup. However, this definition is not useful for semigroups S with zero since for every such S, we have $\sim_o = S \times S$. Note that $\sim_o$ is the largest equivalence contained in $\sim_l$ and that $\sim_p \subseteq \sim_o$ since if $a = uv$ and $b = vu$, then $au = ub$ and $bv = va$.

The aim of this paper is to introduce a new definition of conjugacy in an arbitrary semigroup, avoiding the problems of the notions listed above. (That is, $\sim_l$ is not symmetric; both $\sim_l$ and $\sim_o$ reduce to the universal relation in semigroups with zero; and $\sim_p$ is not transitive and so it requires taking the transitive closure.) Our conjugacy will be an equivalence relation $\sim_c$ on any semigroup S, it will not reduce to the universal relation even when S has a zero, and it will be such that $\sim_c \subseteq \sim_o \subseteq \sim_l$ in every semigroup S,