Journal of Algebra 403 (2014) 135–153

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Varieties of completely decomposable forms and their secants ☆

Hirotachi Abo

Department of Mathematics, University of Idaho, Moscow, ID 83844, USA

ARTICLE INFO

Article history:
Received 12 March 2012
Available online 25 January 2014
Communicated by Steven Dale Cutkosky

Keywords:
Secant variety
Varieties of completely decomposable forms
Generic rank
Waring's problem

ABSTRACT

This paper is devoted to the study of higher secant varieties of varieties of completely decomposable forms. The main goal is to develop methods to inductively verify the non-defectivity of such secant varieties. As an application of these methods, we will establish the existence of large families of non-defective secant varieties of "small" varieties of completely decomposable forms.

1. Introduction

In 1770, E. Waring suggested the problem of finding a positive integer s such that every positive integer can be written as the sum of at most s dth powers of positive integers. In 1909, this problem was solved affirmatively by Hilbert. There is a polynomial version of this problem, which asks, "What is the smallest integer s such that a general d-form in $(n+1)$ variables is expressible as the sum of s dth powers of linear forms?" This problem is often called *Waring's problem for polynomials*. Waring's problem for polynomials remained unsolved for many years, but was completed in a series of papers [3,2,1] by Alexander and Hirschowitz about 15 years ago. In this paper, we explore a variation of Waring's problem for polynomials, which will be described below.

☆ The author is partly supported by NSF grant DMS-0901816.
E-mail address: abo@uidaho.edu.

0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2013.12.027

Let $\Bbbk$ be an algebraically closed field of characteristic 0, let $R = \Bbbk[x_0, \dots, x_n]$, and let R_d be the degree-d piece of R. If a d-form can be written as a product of linear forms, we say that the d-form is *completely decomposable*. Note that R_d is spanned by the monomials of the form $x_0^{d_0} x_1^{d_1} \cdots x_n^{d_n}$ with non-negative integers $d_0, \dots, d_n$ such that $d = d_0 + \cdots + d_n$. In particular, the completely decomposable d-forms form a spanning set for R_d. In other words, every d-form is expressible as a linear combination of completely decomposable d-forms. The minimum number of completely decomposable d-forms that sum up to a d-form is called the *rank* of the d-form. It is worth noting that the rank of a d-form with respect to completely decomposable d-forms is analogous in conception to the tensor rank, the symmetric tensor rank, and the skew-symmetric tensor rank, which are important in areas as diverse as multilinear algebra, algebraic geometry, algebraic statistics, and complexity theory. Please see the recent paper [6] by E. Carlini, N. Grieve, and L. Oeding that contains an extensive list of references related to the rank and the "border" rank of elementary tensors.

The problem we will discuss in the present paper is to find the least positive integer s such that a general d-form has rank less than or equal to s, which can be viewed as a variant of the above-mentioned Waring problem for polynomials. In this paper, we will consider this problem from an algebro-geometric point of view.

The rank-one d-forms in R_d (i.e., the completely decomposable d-forms in R_d), as we shall see in Section 2.2, form a dn-dimensional variety of $\mathbb{P}R_d$. This variety is called the *variety of completely decomposable d-forms* and denoted by $\mathrm{Split}_d(\mathbb{P}^n)$. A rank s d-form is then constructed as a point on a *secant* $(s-1)$*-plane* to $\mathrm{Split}_d(\mathbb{P}^n)$, i.e., the linear subspace spanned by s linearly independent points of $\mathrm{Split}_d(\mathbb{P}^n)$. The Zariski closure of the locus of proper secant $(s-1)$-planes is a projective variety of $\mathbb{P}R_d$ called the sth *secant variety* of $\mathrm{Split}_d(\mathbb{P}^n)$, which we denote by $\sigma_s(\mathrm{Split}_d(\mathbb{P}^n))$.

By the construction of $\sigma_s(\mathrm{Split}_d(\mathbb{P}^n))$, we can easily see that finding the smallest positive integer s such that a general d-form has rank $\leqslant s$ is equivalent to finding the "generic rank" of $\mathbb{P}R_d$ with respect to $\mathrm{Split}_d(\mathbb{P}^n)$, i.e., the smallest positive integer s such that $\sigma_s(\mathrm{Split}_d(\mathbb{P}^n))$ coincides with $\mathbb{P}R_d$. One can show that the dimension of $\sigma_s(\mathrm{Split}_d(\mathbb{P}^n))$ is expected to be the smallest integer among the two positive integers $s \cdot (dn+1) - 1$ and $\binom{n+d}{d} - 1$ simply by counting parameters. Thus the expected generic rank of $\mathbb{P}R_d$ with respect to $\mathrm{Split}_d(\mathbb{P}^n)$ is the smallest positive integer that is greater than or equal to the quotient when dividing $\binom{n+d}{d}$ by $dn+1$.

As it shall be discussed in a later paragraph of this section, there are secant varieties of varieties of completely decomposable forms, which do not have the expected dimension (such secant varieties are often said to be *defective*). If $\sigma_s(\mathrm{Split}_d(\mathbb{P}^n))$ is defective for some s, then $\mathbb{P}R_d$ could have larger generic rank with respect to $\mathrm{Split}_d(\mathbb{P}^n)$ than expected. Therefore, classifying the defective secant varieties of varieties of completely decomposable forms is vital for solving our version of Waring's problem.

A crucial step toward the classification of defective secant varieties of varieties of completely decomposable forms is the development of theoretical tools which can be used to establish the existence of large families of non-defective secant varieties to $\mathrm{Split}_d(\mathbb{P}^n)$.